Appendices

Twelve scholars, who have dedicated a significant portion of their academic lives to teaching and the study of teaching techniques, have agreed to contribute Appendices to this new edition of *How to Teach Mathematics*. It is safe to say that most of these good people disagree with parts of, or in some cases much of, what I have to say. Others agree with me, but think that I am not sufficiently conservative or sufficiently liberal (depending on the point in question). Of course there is no "correct" position on any teaching issue.

I hope that these Appendices illustrate the precept that there are many ways, and many styles, of thinking about good teaching. Certainly the variety of ideas offered here enriches what this book has to offer.

The Irrelevance of Calculus Reform: Ruminations of a Sage-on-the-Stage[1]

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It is not an appealing task to rain on the enthusiastic Calculus Reform Parade. Indeed most of the efforts described on these pages have been undertaken by competent mathematics teachers filled with good will and good intentions. Most projects have succeeded in some sense or other. Who could possibly object?

Let me briefly state the issues I wish to address. Then in the remainder of the article I shall amplify my concerns.

First, the real problem in undergraduate education (not just mathematics) is that students are not studying enough; only grade inflation (i.e. lowered standards) allows them to pass. Second, by not addressing directly the failure to study, calculus reform is, in the long run, irrelevant. Third, attention to the amount of time devoted to homework should be a major concern of each teacher. Fourth, institutions of higher learning undermine high standards by: **(1)** supporting large lectures, **(2)** requiring computerized student evaluations, and **(3)** never considering homework policies in any personnel decisions. Fifth, there are major institutional projects impeding high standards; these often come masked with names suggesting support for high standards: Total Quality Management, Outcome Based Education, the proposed NCTM assessment Standards, the mathematics teaching standards proposed by the National Board for Professional Teaching Standards, etc.

First, STUDENTS ARE NOT STUDYING. Most of us suspect this subconsciously if not consciously. A major survey of undergraduate study habits at my own university (Penn State) suggests that 2/3 of our students study less than 15 hours a week. The Pace Report by the Center for the Study of Evaluation documents a similar phenomenon at the national level. Now if a standard load is 15 semester hours, then the old rule of thumb (2 hours outside class for each hour inside) suggests an average study time of 30 hours. Very few students are studying close to 30 hours a week.

Assuming this first point, I would argue that it is symptomatic of low academic standards and of moral abdication. We have *failed* to teach our students the virtue of hard work. I grant that a hypothetical student could study for 30 hours a week and learn little; so getting students to study is not our ultimate object. Our object is educated graduates. However we can be fairly certain that the non-studying majority is not getting educated adequately.

Given this major problem, how does calculus reform respond? At best, calculus reform addresses this question indirectly. It extols the joys of group learning, sings the praises of thrilling technological innovations such as computerized laboratories, and preaches the virtues of open-ended problems and term

[1]This article appeared, under the same title, in the January, 1995 issue of *UME Trends*. It is reproduced here with the permission both of the publisher and the author.

projects. Now each of these items has merit. Each is especially likely to interest a teacher who has become bored with standard methods. Each may affect the amount of time students study; however I have never seen any of these innovations touted primarily for its substantial positive effect on high standards and study habits. My impression is that calculus reformers believe they are making calculus so interesting that everyone will, in the natural course of things, just work a lot harder and do better.

Increased Achievement

I would welcome evidence that increased achievement is a substantial effect of many of these reforms. I have yet to be reassured. Mostly I see remarks like those of Tom Tucker in *Priming the Calculus Pump*: "...in every case we know where students from experimental sections took a common final exam with the rest of the calculus students, the experimental students did just as well or better." Of group learning, Davidson and Kroll report: "Less than half of the studies comparing small-group and traditional methods of mathematics instruction have shown a significant difference in student achievement; but when significant differences have been found they have almost always favored the small-group procedure." Surely we could hope for much more than this from expensive pilot projects. I suggest that part of the reason for the barely discernible improvement is inadequate attention to standards and study habits.

Furthermore, the technology aspect of calculus reform is especially disturbing. Many of our students have pitiful skills in arithmetic, algebra and trigonometry. While our brightest may gain much from MATHEMATICA projects, we must be vigilant lest the $B-$ or $C+$ students replace basic math skills with button pushing. (This issue has significant philosophical roots and is one aspect of the debate between the Artificial Intelligence enthusiasts and those of us who regard computers as pencils with power steering.)

If you suggest that the computer has obviated the need for facility in arithmetic, algebra and trig, then please provide firm evidence for your views. Part of the reason the New Math floundered was its unyielding contempt for anything remotely resembling "mere rote learning." If "mere rote learning" were renamed "essential drill," we might give it the respect it deserves.

For brevity, I shall combine my third and fourth points. If calculus reform is irrelevant, then what should be done individually and institutionally to enhance the education of our students?

Each instructor should think a lot about how to increase the amount of work students do outside of class. Clearly both traditionalists and reformers can address this problem directly. Institutionally we assess teaching in ways that either ignore the homework question totally or work against it. How many times have you participated in a promotion or tenure decision where homework policies were discussed as a major component of teaching effectiveness? The correct answer is zero! Indeed, what you probably discussed were computerized scores from student evaluations. ("This candidate only scores 3.82 on the Overall Instructor Rating against the college norm of 4.91.") I contend that *computerized* student evaluations are a menace to high standards and demanding teaching. Is it not human nature for students to prefer teachers who make life easy for them? The temptation, almost always unspoken, lurks in the background; I'll you're a good student if you pretend I'm a good teacher.

I also believe that small classes are important. The strong personal relationships between teacher and students (which are impossible between a teacher and 400 students) can be used to support active homework policies.

External Pressures

Finally I wish to draw attention to some of the external pressures against high standards posed by movements and institutions with only the most benign intentions. ("If I knew for a certainty that a man was coming to my house wither the conscious design of doing me good, I should run for my life ..." Henry David Thoreau.)

To begin with we have the newly proposed NCTM Assessment Standards for School Mathematics (to date, I have only seen a draft copy). This 243 page document is *horrifying*. It extols almost any variant of politically correct, socially aware mathematics no matter how insipid while heaping contempt on traditional forms of teaching. It bashes current forms of standardized testing not for being inadequately objective measures but rather because they are objective measures at all. If the NCTM Assessment Standards are eventually published in more or less the form I have seen, they will produce entering college students even more poorly prepared than the ones you have now. If you don't believe me, take a look.

The same consciousness-raising mentality that animates the draft of NCTM Assessment Standards also afflicts the Total Quality Management partisans and the Outcome Based Educators. Again and again, programs with names suggesting high standards show little concern for high standards and much concern for a visionary social agenda. One cannot have high standards and egalitarianism as co-equal priorities: They work at such cross-purposes that one must dominate.

In conclusion, I would like to suggest that we Sages-on-the-Stage are not opposed to progress, despite all rumors to the contrary. We are anxious to encourage and support programs that enhance achievement. Indeed most of us hope that all you Guides-on-the-Side will really be able to respond convincingly to the issues raised in this article.

Mathematical Content

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1. Introduction. Steven Krantz has given us a lot to think about in both his first and second editions. Fortunately, he has left a little to be said. Two of my main concerns are the mathematics which is taught, and the standards which need to be set. In a time when rapid changes are being made, both in the curriculum and in teaching methods, it is important to make sure that the essential parts of a mathematics program continue to upheld. That is the reason for my concern about content and standards. As will become clear, there are good reasons to be concerned about the mathematics which is being taught. Let me start with an example where I was not as careful about this as I should have been.

About 25 years ago, my wife was taking a course in statistics in the Sociology Department. Pocket calculators were very expensive, and the students needed to compute some squares roots, so the professor announced a review to teach how to do this. I called and asked him how he was going to teach this. This method was based on

$$(10a + b)^2 = 100a^2 + (20a + b)b,$$

although this is not how he described it to me. I explained an alternate method, of guessing an approximate root, dividing and averaging to find a better approximation. The next day he cancelled the review, thanked the unknown student who had a friend in the Mathematics Department, and passed out notes on an easy way to compute square roots.

There was a weekly laboratory session. That week one student complained to the teaching assistant about having to learn a new way to take square roots. She knew the other method, and did not want to have to learn a new one. The teaching assistant agreed with her, and said he had learned the other method as a freshman student of engineering at Washington University. He never put together the name of one student, Elizabeth Askey, with the name of the young instructor who had taught him this. It was reassuring to know that he remember this, but I should have thought a bit about different ways to teach how to compute square roots.

2. Learning trigonometry. One of my concerns is the mathematics which prospective teachers learn. One subject which high school teachers need to know very well is trigonometry. Here is what I think high school teachers should know about trigonometry.

The first idea is similarity of triangles. For right triangles, similarity allows the trigonometric functions to be defined as real valued functions.

The Pythagorean theorem gives

$$\sin^2\theta + \cos^2\theta = 1$$

and the other related identities. These, along with definitions and elementary algebra, show that any two of the six trigonometric functions are related.

The next idea is that an arbitrary triangle can be decomposed into two right triangles. This decomposition is usually used to prove the law of sines and the law of cosines. It can also be used to prove the addition formulas for $\sin(\theta+\varphi)$ and $\cos(\theta+\varphi)$.

Consider the following picture.

The area of the large triangle is

$$\frac{1}{2}ab\sin(\theta+\varphi).$$

It is also the sum of the areas of the two smaller triangles, so is

$$\begin{aligned} &\frac{1}{2}ah\sin\theta + \frac{1}{2}bh\sin\theta \\ &= \quad \frac{1}{2}ab[\sin\theta\cos\varphi + \cos\theta\sin\varphi] \end{aligned}$$

I learned this argument from a forthcoming book on trigonometry written by I. M. Gelfand and M. Saul, [9]. It appears elsewhere, but is not well-known.

A similar argument using the law of cosines leads directly to the addition formula for $\cos(\theta+\varphi)$. I would assign this as a problem. The addition formulas are so important that a different type of proof should be given. Another one which uses an important idea is to use the invariance of the unit circle under rotation. Both of these proofs should be given in what used to be a course in trigonometry. Now, in our high schools, this is usually done in precalculus.

High school teachers should know other proofs. A simple one comes from

$$e^{i(\theta+\varphi)} = e^{i\theta}\cdot e^{i\varphi}$$

and

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This can be done in detail in advanced calculus, and can be done in calculus if multiplication of power series and convergence of series of complex numbers are done. I think the first should be done, and the second sketched, since both are very useful topics.

There are also derivations of the addition formulas for $\sin(\theta+\varphi)$ and $\cos(\theta+\varphi)$ from the differential equation

$$y'' + y = 0,$$

and from the multiplication of two rotation matrices in the plane. I do these derivations in the courses where linear differential equations and rotation matrices are treated.

There should be many applications of trigonometry in addition to the standard problems of solving for missing parts of triangles. One use of the addition formula for $\sin(\theta+\varphi)$ is to show that

$$a\sin\theta + b\cos\theta = (a^2+b^2)^{1/2}\sin(\theta+\varphi).$$

The addition formula for $\tan(\theta-\varphi)$ gives the angle between two lines in terms of their slope. I would ask students to find the angle between two lines whose slopes are known. It is not too hard to see that $\tan(\theta-\varphi)$ is useful, when $m_1 = \tan\theta$, $m_2 = \tan\varphi$. The formula for $\tan(\theta-\varphi)$ would not have been derived, so something would have to be done. By this time in a course, students should have learned that when trying to simplifying a trigonometric expression which contains $\tan u$, it is natural to replace $\tan u$ by $\sin u/\cos u$. From there on, it is a matter of using the addition formulas for $\sin(\theta-\varphi)$ and $\cos(\theta-\varphi)$, and then trying to find a way to simplify the resulting expression.

Tony Gardiner has a very nice set of notes titled "Recurring Themes in School Mathematics" [8]. He makes the point that meaningful simplification is one of the goals in life one should start to learn in school. The necessary simplification needed above is a good instance of meaningful simplification.

Here is a use of a consequence of the addition formula to an interesting problem in algebra. The problem is to find the polynomial $p_n(x)$ which satisfies

$$|p_n(x)| \le 1,\ -1 \le x \le 1, \tag{2.1}$$

and takes on the largest value for each fixed $x > 1$. This is a result of Chebyshev. The related problem of finding the polynomial of degree n satisfying (2.1) which has the largest derivative at $x = 1$ was solved by Mendeleev when $n = 2$, and by A. Markoff for general n. See Boas [6] for a nice treatment.

This problem can be started as follows. First, use the addition formula to show that

$$\cos 2\theta = 2\cos^2\theta - 1.$$

Graph both sides as a function of θ, and graph the right hand side in x, with $x = \cos\theta$. Then, have the students derive the similar formulas:

$$\cos 3\theta = 4x^3 - 3x$$

and

$$\cos 4\theta = 8x^4 - 8x^2 + 1,$$

and graph these as functions of x for $-1 \leq x \leq 1$, and for $0 \leq x \leq 2$. When $-1 \leq x \leq 1$, these polynomials satisfy

$$|p(x)| \leq 1, \ -1 \leq x \leq 1.$$

Students should be asked to explain why this is true. A graph on a graphing calculator is not sufficient to explain this without a lot of extra work. There is a more fundamental reason, which also allows polynomials of higher degree to be generated. These reasons are:

$$\cos n\theta = T_n(\cos\theta)$$

is a polynomial of degree n in $\cos\theta = x$, and

$$|\cos n\theta| \leq 1,$$

so

$$|T_n(x)| \leq 1.$$

It is not too hard to see that if $p_2(x)$ is a polynomial of degree 2, and $|p_2(x)| \leq 1$, $-1 \leq x \leq 1$, then

$$|p_2(x)| \leq T_2(x) \quad \text{for each fixed} \quad x > 1.$$

A slightly more complicated argument, which is mainly counting zeros of $p_n(x) - T_n(x)$, can be given to show that

$$|p_n(x)| \leq T_n(x) \quad \text{for any fixed} \quad x > 1$$

when $|p_n(x)| \leq 1$ for $-1 \leq x \leq 1$.

To show that $T_n(x)$ is a polynomial of degree n in x, one can use

$$\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta$$

or

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

Either of these identities can be proven from the addition formula for $\cos(\theta+\varphi)$.

One problem about teaching this material to prospective high school teachers is that most of them will have learned trigonometry in high school and not take it in college. One course in which this can be done is a course on proofs. J. Rotman has written a book [14] for such a course which contains many of the necessary parts of trigonometry. He also included the rational parametrization of the circle by

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2},$$

and use of these formulas to obtain Pythagorean triples. He pointed out that this parametrization can be used to verify many stated trigonometric identities. For example, he used these formulas to prove that

$$\frac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta} = \sec\theta+\tan\theta,$$

He then quotes Silverman and Tate [15]:

> "If they had told you this in high school, the whole business of trigonometric identities would have been a trivial exercise in algebra!"

This is not true, for what should be taught when students are learning how to prove an identity is to teach how to do meaningful simplification. See [4] for comments on this, and an example of how not to teach trigonometric identities.

If high school teachers have not learned what the essence of trigonometry is, one could hope they would see this in the textbooks they use, or in articles in "Mathematics Teacher". There were books which had very nice treatments of trigonometry. One such book is [9]. However, many books have a treatment not too dissimilar to another one mentioned in [4], so one can not count on textbooks.

Here is a brief excerpt from an article in Mathematics Teacher, [12].

> The trigonometry teacher can use the graphing calculator in teaching identities. These equations can be used:
>
> $$\begin{aligned} y_1 &= \sin(2x) \\ y_2 &= 2\sin x \\ y_3 &= 2\sin x\cos x \end{aligned}$$
>
> Have students graph y_1 and y_2. Two appear! Next have students graph y_1 and y_3. The students are now learning identities not by the rote method of pencil and paper but by experiencing and *seeing* an identity.

This appeared in a section titled "Sharing Teaching Ideas". The last sentence incensed me, so I wrote to two officers of NCTM to suggest that they write an article pointing out the improper use of the word "learning". Nothing appeared, either as an article or in the "Readers Reflections" section.

Hung-Hsi Wu wrote a paper about mathematics education [16], and commented on the quotation above. He noted that nothing was said about proving this identity. Then he continued with: "Now if the authors had said that 'in addition to proving the identity $\sin 2x = 2\sin x \cos x$, using the graphing capability of a calculator can reinforce students' confidence in the abstract argument,' we could have applauded them for making skillful use of technology in the service of mathematics."

Jeremy Kilpatrick [11] wrote a rejoinder to Wu's article. Here is one paragraph.

"[Wu] also hits some inappropriate targets. For example, he castigates two high school teachers [12] writing in the *Mathematics Teacher* for their attempt to help students see the functions involved in a trigonometric identity before establishing its validity. Quite apart from whether every article in an official journal of an organization promoting reform must reflect reform views, one can reasonably ask whether seeing the graphs of these functions alone might not help students understand the identity. And how can Wu be so certain that teachers who are having students use graphing calculators are neglecting proof just because it is not mentioned in the article?"

Unfortunately, I feel Kilpatrick has misread this article, and the authors meant exactly what they wrote when they wrote: "The students are now learning identities not by the rote method of pencil and paper but by experience and *seeing* an identity."

The lack of a proof of the double angle formula is not the only drawback in [12]. The graphical picture gives no idea why the identity is true, so no clue about how to extend it. Also, it gives the false picture of mathematics that formulas come out of the air, or are given by the teacher, with the student not expected to understand how anyone could dream up such a formula. Important formulas are almost never found in such a way.

After waiting almost a year and a half, I wrote a short note about [12] and submitted it to "The Mathematics Teacher". This note included the decomposition proof above in the special case when $a = b$, and suggested that to see if students had learned something they should be able to derive the addition formula for $\sin(\theta + \varphi)$ by a similar argument. This note was turned down with the following report from a panel member.

> "This adds nothing to the Sharing Teaching Ideas article
> – it is just a list of complaints."

3. Recurring themes in school mathematics. Tony Gardiner's booklet "Recurring Themes in School Mathematics" was mentioned earlier. Here is an example of a recurring topic which he does not treat.

In late elementary school, students should learn about the connection between decimals and rational numbers. They need to do some calculations from rational numbers to decimals, to see why the resulting decimal either terminates or repeats. The number $5/17$ is a good example to give, but not as a first example. In a course for prospective elementary school teachers, I gave this as homework, with the students to say whether the decimal expansion repeated or not, and to explain their answer. They were to do the calculations by hand. A substantial minority said it did not repeat, and others said it did but could not explain why other than to say that it did because of their calculation. A few were able to explain why it had it, since the remainders would eventually repeat.

In addition to this direct problem, the inverse problem of going from a repeating decimal to a rational number should be considered. In [8], Gardiner made the point that inverse operations are frequently more important than the associated direct operations. That is true here when one considers repeating decimals rather than decimal approximations to rational numbers. The reason for the importance is that when this operation is done correctly, it leads to a method of summing a general geometric series.

The right way to do this change from a repeating decimal to a rational number is to call the repeating decimal x, multiply by the appropriate power of 10 to move the digits over one period, and then subtract to reduce to problem to a finite decimal expansion. Currently, there is another way which is being used in some textbooks. This is to use pattern matching. In the best version of this

$$\frac{1}{9} = .\overline{1}$$

and

$$\frac{1}{99} = .\overline{01}$$

are extended to

$$\frac{1}{99\cdots 9} = .\overline{0\cdots 01},$$

and so get

$$.\overline{45} = 45(.\overline{01}) = \frac{45}{99}.$$

This argument does not work directly when the repeating part does not start immediately, but there are ways around this problem. See [1], [13] for an illustration of what is currently being written. The real problem with this method is not that it is usually just given as a rule to follow without any reasons given, but that it is a dead-end method.

In middle school, after exponents have been introduced, students should be given the problem of how many grains of rice would be given if one is given for the first square of a chessboard, 2 for the second square, 4 for the third, and doubling each time until 2^{63} grains are given for the 64th square. The classical story is adapted nicely by David Barry [5]. On the last page, he illustrates approximately what the number of grains of rice would fill, or be equivalent to. He starts with 2 grains on the first square. 2^8 fills a teaspoon, 2^{16} - a bowl, 2^{24} - a wheelbarrow, 2^{32} - a festival hall full to the roof, 2^{40} - a palace with 256 rooms, 2^{48} - the World Trade Center, 2^{54} would cover Manhattan island 7 seven stories deep in rice, and 2^{64} would make a mound as large as Mt. Kilimanjaro. He concludes by saying that all 64 squares together would cover all of India knee deep in rice.

Students should be asked how much rice that is, i.e. sum the series $2 + 2^2 + \cdots + 2^{64}$. The use of different measurements for the amount of rice is nice, and needed in the United States. In the Third International Math. and Science Study, and in the Second International Math. Study, our eighth grade students did most poorly on measurement problems, as measured by how well our students did relative to the international average. We have to teach our prospective elementary and middle school teachers both the exact calculations which lead to $2^{65}-2$, and the various approximate measures, if we expect them to teach this to our elementary and middle school children.

Another delightful children's book which can be used with late elementary and middle school children and in some of our college courses is Anno's "Socrates and the Three Little Pigs", [2]. I have used this successfully in a summer school for students between fourth and fifth grade, and in both our arithmetic course for prospective elementary school teachers and our undergraduate combinatorics course.

In high school, students should learn how to divide 1 by $1-r$, getting

$$\frac{1}{1-r} = 1 + r + \cdots + r^n + \frac{r^{n+1}}{1-r}$$

and how to derive

$$1 + r + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$$

by giving the left hand side a name, multiplying by r and subtracting. This should then be done formally for the infinite series, and then tied up with the repeating decimals from years earlier.

At present, since few calculus students know how to sum a geometric series, this needs to be done in calculus. I do it early, and start with repeating decimals. Unfortunately, even this is unknown by a large majority of our students. An increasing number of those who know that $.\overline{47}$ is $47/99$ only know it by the rule mentioned earlier of taking 47 and dividing it by 99. A good way to check if this is all they know is to ask them what rational number corresponds to

$$.5\overline{47}.$$

To show that the geometric series is useful, I use it to differentiate x^n, $n = 2, 3, \cdots$, and later x^r where r is rational. I also use it to integrate x^n from 0 to a, first when n is a positive integer, and later when n is a positive rational number. See [3] for this and some references.

When students have seen special cases of the geometric series for abut six years before they see the general case, and have seen a number of different applications of it, they will have an easier time understanding the ratio test for convergence of series, and be able to use the geometric series to find error estimates for the remainder of some important power series.

A delightful problem on the geometric series was given by Gelfand and Shen [10, §45]. They introduce infinite geometric series by means of the paradox of Achilles and the turtle, with Achilles running 10 times as fast as the turtle. After giving four methods to sum this series, and getting the general formula

$$1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q},$$

they take the case when Achilles runs ten times slower than the turtle. This gives the absurd answer

$$1 + 10 + 100 + 1000 + \cdots = \frac{1}{1-10} = -\frac{1}{9}.$$

Then they give the following problem.

> Problem. Is it possible to give a reasonable interpretation of the (absurd) statement "Achilles will meet the turtle after running $-1/9$ meters"?
> Hint. Yes, it is.

4. Large lectures and teaching assistants. There are problems which arise when teaching a large lecture with a number of teaching assistants. Krantz has mentioned many of them. At the University of Wisconsin-Madison, first year calculus lectures meet three days a week, and there are discussion sections which meet twice a week. The size of the discussion sections is bounded, so the more students in the lecture, the more teaching assistants. This means that it is possible to give and grade exams where students have to write complete solutions. I always do this. I write the first exam. The remaining exams are written by the teaching assistants, usually in pairs. During our weekly meetings, we discuss the exams after a final draft has been written, and worked by the writer(s) of the exam. Working the exam in detail is important, both to catch little problems which show up when complete solutions are written, and so that the teaching assistants get some idea how long it takes them to work an exam in comparison to how long students will take. All of us have a chance to make comments and suggest changes. I try to say as little as possible, since this is very good experience for the teaching assistants. However, if the exam is too long, which is a common problem the first time an exam is written, I am able to get it modified appropriately. One question which regularly comes up is what is a specific question really asking, and is this the best way to ask it? I try to mention something about this when the first exam is being discussed, and that leads others to ask this question later.

A few of my colleagues have had one teaching assistant who essentially told the students what the questions are, and so do not let any of the assistants see the exam before it is given. I have not had this problem, probably because the teaching assistants know they will be writing one of the exams and do not want this to happen to an exam they help write.

I disagree strongly with one of Krantz's recommendations. He wrote that a teacher should *never* penalize a student for being honest, so that a student who comes and says the points were added incorrectly should just be sent home with a little praise for being so perceptive. Consider what happens when you are in a small grocery store without a fancy check-out machine, and the clerk gives you incorrect change. If you are not given enough, you tell the clerk. I hope you also tell the clerk when you are given too much. I would be insulted if the clerk said to forget it and praised me for being honest. In forty years of teaching, I can remember three students who came and said a problem was misgraded, and too many points were given, I regraded the problem, and gave the correct grade. I also remembered who had come, and made sure that the few points did not change the final grade. There have been a few other instances where scores were added incorrectly. Those were also changed.

When a student in a large lecture comes to ask about possible misgrading, my answer is that I will look at the grading of the full exam, not just that question.

My solution to the problem of a fixed formula for adding grades to get a course grade is to say that any one exam which is out of line with the others counts less, unless it is a higher grade on the final. Doing this both ways can be important in the first semester calculus with students who took a calculus course in high school, and think they know more than they do. They may do reasonably well on the first exam, even without studying, but after that do worse, and on the final show they really did not have a firm command of the early material in the sense of being able to use it in more complicated settings. They probably should fail the course even if the first exam pulls an otherwise failing grade up to D.

Both of the last two examples are illustrations of maintaining standards. Here is another instance.

A few years ago, after an eight hour committee meeting on Saturday, I went to an afternoon party at a local jewelry store. I was tired and a bit surprised when a man approached and asked if I were Professor Askey. After answering "yes", he said he had me for calculus. Bored, I said: "You and a few thousand others". He said I kicked him out of class. My reply was: "That was a while ago". He said I kicked him out for smoking. That was a long time ago. He continued, saying that he was just back from Vietnam and had a chip on his shoulder. I had started the first lecture by saying that smoking was not allowed in class. He decided to challenge me, and lit up a cigarette. I kicked him out. He went back to his room, and started to think what he was doing with his life. He decided he did not like what he was doing. He liked mathematics, so became a mathematics major. He changed to philosophy in his last year, now runs a jewelry store in another city. He said he had wanted to thank me for changing his life. We all hope to have an impact on some students, but I had not expected this to come directly from kicking a student out of class. Keeping standards is important.

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Personal Thoughts on Mature Teaching

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This book is directed toward new teachers, but much of the advice can be summarized in two fundamental attitudes that are essential for all teachers: enthusiasm for the subject and concern for the students. These attitudes got me through all manner of pedagogical mistakes in my early years as a teacher. As I have matured, I have found that my early enthusiasm and concern are still important, but they are not enough. These are my thoughts about what comes next for those who have been teaching for several years, who know the basics of what to do and what not to do in the classroom, who already are good teachers.

I have always known that enthusiasm is critical. This intuition was confirmed in "The Quest for Excellence in University Teaching" [**1**] which lists enthusiasm as the most frequently cited attribute of excellent teachers, whether one asks students, colleagues, or administrators. This fact echoes what I have experienced in my own classes and those of others. Enthusiasm is infectious. Even when my students do not share it, they respect it.

My first responsibility in preparing for any class is to find something that I consider interesting and exciting, that I want to show and to share. The fact that I find the material extremely elementary is no excuse, nor that I have taught this course twenty times before, nor even that I was not involved in constructing the syllabus which to me seems disjointed and pointless. As a good teacher, I look for new and interesting examples and illustrations, even in the most elementary mathematics. I search for fresh insights and approaches to what I have taught before and never rely on last year's notes. I seek the threads that tie together the pieces of the syllabus.

One danger of enthusiasm is to become overly ambitious for your class. This is an error that I committed frequently in my early years of teaching; many new teachers do. Even after more than twenty years at the head of the classroom, I still make this mistake more than I would like to admit. It is a forgivable error that has the potential for good—I have seen it stretch students far beyond what they thought they were capable of—provided it is accompanied by the second critical attitude, concern for the student.

Steve Krantz has said a lot of good things about the basic components of concern for your students: Respect them and their questions; get to know at least some of them as people; honor the commitment to have times when you are available to them; learn to be sympathetic, receptive, and patient; be aware of your audience; care. It is common for new instructors to have a rough time the first semester they are in charge of their own class. The assignments and exams are often much too hard. They do not yet have an adequate repertoire of examples and alternative explanations to fall back on when difficulties are encountered. New instructors can be oblivious to some of the pitfalls that students will encounter. But concern for the students will overcome these. I have been privileged to see the student evaluations of several teachers who before that semester had never been responsible for their own class but who had both enthusiasm and concern. Many of the evaluations were critical, describing the things that had gone wrong during the semester. But all of the critical evaluations ended with words to the effect "but she is going to be a really good teacher." They were right.

There comes a time when the naive enthusiasm and concern of the new teacher is not enough. For me it arrived after I had developed confidence in my own ability to teach. I knew the difficulties that students would have and I carefully set out my warnings. I had laid up a stock of illuminating examples that I had seen work for students who were confused. I knew the important concepts that must be stressed and how to tie them together effectively. I was writing assignments that stretched my students, but not beyond their abilities. Exam results were coming in that were not quite what I wanted, but that were adequate. And so I dared to probe a little deeper, to give the students an opportunity to show off their understanding. What I discovered horrified me.

I believe that this moment comes to most of us. It might happen when talking with a student in your office. It might be the result of asking something on a test that goes beyond formulaic responses and requires students to draw on their knowledge to synthesize an answer to a question that is not quite like any they have seen before. It might arrive in a written report in which, for the first time, they are required not just to give the answer—which you know they can find—but to explain how they got that answer. Where you had thought there was understanding, you discover confusion. It seems that they have learned nothing and that your only recourse is to go back to the beginning and start all over again.

This is a moment of crisis—in its literal sense of decision—for the good teacher. I have known those who have resolved it by blaming the students, either their unwillingness to work or the poor preparation that they received from others. I have known those who have resolved it by resigning themselves to the belief that what happens in their classrooms for the majority of students will never be more than a superficial and temporary acquisition of what is needed to pass the course. They decide to focus on those few students who are most like themselves.

Either of these responses marks the beginning of a loss of concern. The concerned response is to seize this opportunity to recognize the depths of our own ignorance about what actually happens in the classroom and how students learn. If we are good teachers, we will force ourselves to begin exploring the murky waters of pedagogical theory and educational psychology.

This is our opportunity to honestly face what does and does not work in our own teaching and to discover what others have done, to experiment with their ideas and techniques. We have the opportunity to build up a body of insights and ways of capitalizing on those insights. It is my belief that this is a dynamic process that never ends. The truths that I now know about what happens in my classroom are subject to articulation, refinement, and elaboration. Priorities will shift. Techniques for achieving those priorities will always be under trial. Being a good teacher has become a highly personal quest.

Ironically, the great truths that I and others discover are never new. Reading through the literature of education, we find that each hard-won truth was known to the previous generation and to generations before that. But that does not eliminate the need to rediscover these truths for ourselves, individually. We are now seekers after wisdom, and there is no quick route to it. We can learn from the wisdom that others have accumulated, but it is not our own wisdom until we have sifted it, found our own ways of applying it to our teaching, watched it succeed and watched it fail, and made its lessons part of our own story.

I would like to offer some of my own truths. They are neither new nor will they solve your problems. These are not answers. These are places to begin asking your own questions.

Teach the Students You Have

The temptation to blame the attitudes of our students or their preparation from previous teachers is very real because it often seems justified. Working to improve teacher training or what happens in our K–12 schools is an important and worthy contribution, but such efforts do not absolve us from the responsibility to face the students before us with all of their imperfections. We must begin by engaging them where they are. This has nothing to do with abandoning standards or watering down the curriculum. On the contrary, nothing will be accomplished unless we are optimistically realistic about our students and their capabilities.

The situation is never as bad as the cynics and pessimists paint it. I start each semester by asking each student about expectations for the class and what they want to get from it. Almost universally, they want to "understand" the mathematics in this course, and they expect to have to work hard. But they are busy people with many demands on their time, and they are uncertain about what it means to understand mathematics or how to go about accomplishing this. I must provide the structure to help them achieve these goals. Also, no matter how well I may think I know the abilities of the students who will be in my class, there is no substitute for diagnostic tests or assignments.

Establish and Communicate Clear Goals

If you do not know where you want the students to be by the end of the semester, there is little chance that they will get there. As you set your goals, you need to be optimistic. I have found that students respond well to the challenge to work hard provided that they believe that your goals are attainable and that the necessary support mechanisms are in place. Unrealistic goals can be dangerous. Once students begin to feel overwhelmed, they adopt a strategy that David Tall [**5**] has called *disjunctive generalization*. They jettison the search for understanding and switch to memorization, regardless of contradictions or inconsistencies, as the safest route to a passing grade.

Writing down your goals for the course is a good exercise for you. Think about the kinds of problems you want students to be able to solve by the mid-point, by the end of the course, a year from now. How important are the ability to apply this mathematics to novel situations? to analyze the components of the theory they have learned? to creatively synthesize these components into creative problem solving? What are the big ideas that they need to be conversant with? What are the connections that they have to be able to make?

If you want to share this list of goals with your students, that is fine. It will help relieve some of their start-of-semester anxiety. But you communicate these goals in how you choose to assess their performance. If you do not measure and reward their progress, then few students will make those goals their own.

Use Assessment Effectively

It is common to base most of the course grade on student performance in two midterm exams and a final. Students are accustomed to exam questions that are a mixture of routine straight-out-of-the-book exercises and a few more challenging problems. They know that they can get a good grade by practicing homework problems until they have the basic patterns memorized. They do not have to be able to answer the most difficult problems. We may say that our goal is for our students to understand the mathematics, but if we use this standard format to assess their knowledge, then we are telling them that what we really care about is whether they can mimic the template solutions quickly and accurately. We are allowing them to bypass understanding.

We want our students to be able to do basic calculations, but we also want them to learn more than this. We would like to be able to replace the standard exam with one that consists of probing, challenging questions that give students a chance to show off what they really know. I know from my own experience that unless I have specifically prepared my students for such a test, the result is abysmal scores and vociferous complaints that the test was unfair; it was not the kind of test my students were expecting. If we have given a challenging test on which students have scored poorly, we are tempted to use a curve to determine grades. This undermines what we were trying to accomplish because it confirms the message that partial credit is good enough and real understanding is not important.

The standard assessment based on two midterms and a final is a trap that leads to one of three unsatisfactory outcomes: Either students see facility with certain well-defined procedures as the goal of the course, or they learn to rely on a curve and so know that they will not be held accountable for any real understanding, or most of the students fail, which not only makes life difficult for the math department but shows that you are not teaching *your* students.

There are other ways of approaching assessment.

Assessment is the carrot and stick that you can use to shape student attitudes and study habits and to communicate what you want students to learn from your course. If you want students to read the text, then give unannounced quizzes on the readings that they were to have done. If you want students to reflect on what they have learned and think about what is happening in the processes that they are mastering, then have them keep a journal that is periodically graded or have them write reports that require this reflective approach. If you want your students to be able to apply their knowledge to unfamiliar situations, then give them problems and projects that require this level of understanding. If you want your students to listen critically to your lectures and be able to distill the main points, then ask them to write these down at the end of the class and hand them in. If you want students to be able to use definitions and theorems correctly and unambiguously, then have them write assignments where this is required. If there are basic skills that you want students to master, then test these skills and set the bar for a passing grade as high as you feel is needed, whether that be 80%, 90%, or 100%.

Assess early; assess often. In my experience, students react positively to this. They appreciate the feedback and direction that it gives them. It reduces the stress of a major examination that counts for a third or more of their grade and for which they are not quite sure what will be expected. Students should never be surprised by what you expect of them. You should be shaping their approach to the class from day one. There is nothing wrong with putting probing, challenging questions on your examinations provided this is nothing new and that students have been given the means to tackle such questions. There lies the nub of the difficulty. Once you know where your students are as they begin the class and you have made clear to them what will be expected, how do you make it possible for them to achieve those goals?

Put Supports in Place

Take your big goals and look at the pieces that have to be in place in order to achieve them. You want your students to be able to tackle and solve an unfamiliar problem. It requires going back through all that they have seen this semester, picking out appropriate ideas, and putting them together in what is for them an original configuration. Do they need to be conversant with the big ideas of the class? Do they need to be able to go back and reread sections of the text? Do they need to be certain about the meaning and significance of certain theorems? Do they need to be able to express their own understandings clearly? If so, then these are things that you should have been emphasizing and assessing all along.

You may need to devote time to how to read a math book. Most students believe that this is not possible, so just testing them on how well they read will not be enough. You may need to spend time talking about what you expect in technical reports. You may need to talk about techniques of problem solving. If students know that these are skills for which they will be evaluated, they will pay attention. If your syllabus is so crammed that there is no space to work on the skills you consider to be important, then there is something wrong with your syllabus.

The most important support that can be put in place is the opportunity to practice the high level skills that you will demand. This should include critical feedback and provision for students to redo the assignment. One of the great drawbacks of the traditional "two midterms and a final" model of assessment is that it re-enforces the perception that the way to get through classes is to take the week before an exam, focus all energy on that course, clear that hurdle, and then forget all that has been learned to clear memory space for the next hurdle. Traditional assessment re-enforces the attitude that success in mathematics is the result of natural aptitude and not something that can be cultivated and developed. We know that mastery comes through the process of attempting and failing and then going back to find the roots of that failure and correcting them. This is an attitude toward learning that we would hope that our students know before they arrive in our classes, but they usually do not. It is our responsibility to lead them through this process until it becomes their own.

I do this through projects and the reports that students write. I have a clear set of criteria about the level of precision and clarity that I expect. Incorrect mathematics, inadequate explanations, and poor writing are all critiqued in the first submitted version which is returned to the students for reworking. On midterm examinations, after I grade and return the test, my students have several days in which to correct the answers that they missed. They can regain some of the points that they lost if they can show that they can now solve that problem. The only complaint I ever received about this policy was from a student who did not want to be forced to think about the test questions he had missed after the exam was over. That complaint has become one of my primary justifications for this policy.

Make Students Active Participants

At one of my first national meetings, I picked up a button that proclaimed, "Math is not a spectator sport." We all learn by doing. I do not know any responsible teachers who do not want students to wrestle with the mathematics that we have explained to them and to practice the higher level thinking skills in which we want them to excel. The problem is that most of our students have no idea how to begin to interact with mathematics in this way. For them, practicing mathematics outside of class is doing the even exercises numbered 2 through 20 at the end of the section, and nothing more. Most of our students come into our classes with no conception of how to begin tackling an unfamiliar problem, or what to do if the first line of attack fails. The answer is not to despair of who they are, but to help them become the students that we want them to be. Our opportunity for shaping the behaviors that we want them to adopt is in the classroom.

Some of the most effective learning I have witnessed has been in group situations in the classroom where a small group of two or three or four students is tackling a challenging and unfamiliar problem, a problem that they will then carry out of the classroom to continue working on. The group dynamics are important. Students working on their own are more likely to freeze and try nothing if the correct approach is not immediately apparent. They are more likely to stick to an unproductive strategy despite its futility. They are less likely to see alternate procedures that might simplify or simply clarify what has worked. As I have seen repeatedly, a small group of students collectively can solve a problem that none of them individually could have worked out. The result is increased confidence and experience in lateral thinking.

Lecture is still one of my tools for teaching, but I have learned that it is most effective when broken up by opportunities for students to actively engage the topic that I am explaining. This includes asking probing questions and giving students time to think about or work on the answers. When there is doubt or hesitancy about the answer, I ask several students to put the answer into their own words. Where there is divided opinion about the correct answer, it is helpful to stop the class and have students discuss it with those around them. A tremendous amount of learning transpires when a student has to explain his or her own understanding to someone else.

Encourage Group Work

Few of the catchwords associated with the reform movement in mathematics instruction have been as controversial as collaborative or cooperative learning. I am a believer for the reasons given above, because I have had students tell me that this is where they learned how to analyze unfamiliar problems, and because of the role that it plays in developing study groups and support networks.

I have held exit interviews with graduating seniors. One of the consistent factors cited as important for their success in college was learning to form and make use of study groups. This was not something that they knew would be helpful when they came to college. Their first study groups grew out of the interactions with classmates that were created by group projects.

There are many approaches to collaborative or cooperative learning. My advice is to talk with others who have experience and then experiment with what feels comfortable to you. The fact that students will brainstorm a problem collectively does not mean that they have to receive a common grade. I have often had each member of the group write up his or her own report. But group grades are useful early in the semester to force students to learn to work together.

Use Technology as Appropriate

Technology is the other controversial catchword of the reform movement. This is the one place where there really is something new under the sun, but reformers are far from unanimous in their understanding of what it means for teaching or how it should be used. My own advice is to stay informed and be willing to experiment with the ideas and approaches that make sense to you. There have been many failures. There also have been many successes. We are still in the early stages of learning how to use this tool.

Advocates of technology say that it enables students to focus on the ideas rather than rote manipulations. Critics assert that it becomes a crutch and that students who are not fluent with fundamental processes are handicapped when approaching higher level problems. My own experience is that both sides are correct, and the important and difficult question is where to draw the line. At what point do we introduce, permit, or encourage the use of technological tools? The answer is highly dependent on our immediate objective. When I want my calculus students to discover the orthogonality of the sine and cosine functions of various frequencies, it is essential to use a computer algebra system to ensure speed and accuracy in performing integration by parts. But before we begin such an exercise, my students will have done a lot of integration by parts problems by hand because I believe that it is a fundamental skill. Its mastery is essential to the appreciation of its consequences.

There are many different ways in which computers or graphing calculators can be incorporated into classes. They can be used to prepare demonstrations, but be wary of presentations that are too slick. Remember that your job is not to entertain but to get students to think. Many math classes now incorporate labs where computers are used for numerical calculations, or to aid in visualization, or for work at the symbolic level. What I have found through painful personal experience is that any laboratory must be tightly integrated into what is happening in the classroom, reinforcing what happened in the previous class and preparing for the next. When the laboratory experience is well thought through, it can provide powerful reinforcement of the lessons that you want to communicate. When it is poorly conceived, it is nothing more than a frustrating waste of time for you and your students.

Be Open to Curricular Reform

When I have thought long and hard about a course—what works and what does not work for my students—I find myself dissatisfied with the traditional syllabus and the available textbooks. The fact that I am not alone is reflected in the myriad reform curricula and textbooks. This is not a monolithic movement. It is characterized by wide diversity. In sorting through what is available, you need to be aware of the goals and priorities of those who developed the materials.

My own primary criterion in choosing or developing curricular materials is to have a driving theme that generates questions that will puzzle, discomfort, and challenge my students. I have come to appreciate that the pure Euclidean ideal of finished mathematics is not appropriate for most teaching. It is a polished surface too slippery for most students to grasp. Imre Lakatos [**3**, p. 140] has gone so far as to accuse Euclid of being "the evil genius particularly for the history of mathematics and for the teaching of mathematics." A kinder assessment is given by David Tall in his "Reflections" on *Advanced Mathematical*

Thinking [**6**],

> This does not remove the need to pass on information in the theorem-proof-application mode, for this is the crowning glory of advanced mathematics. But students need to be assisted through a transition to a stage where they see the necessity and economy of such an approach.

Some of my own thoughts on how to assist students through this transition are expressed in my review of Serge Lang's *Undergraduate Analysis* [**1**] and in "True Grit in Real Analysis" [**2**].

If we want to change what students take from our courses, then we must change what we do. If nothing changes, then nothing changes. We must be realistic about where we and our students start, clear about what we want to accomplish, knowledgeable about how our students learn, and willing to experiment with our teaching to make it as effective as possible.

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Remember the Students

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I didn't like Steve Krantz's first edition of *How to Teach Mathematics* because of its narrow focus on traditional methods.[1] As strange as it may seem, even though he has spent a lot of time talking and learning about different modes of teaching, and even though he has tried to incorporate much of that new knowledge into this book, I think I like this version even less. It seems to me that now he's patronizing people who want to make basic changes in how math is taught.

I think a good place to start to see what I dislike in the book is in Steve Krantz's section 1.6 on lectures. Here are some snippets from that section.

> Those who say that "the use of the lecture as an educational device is outmoded" rationalize their stance, at least in part, by noting that we are dealing with a generation raised on television and computers.
> *SK, p.14*

I don't think the lecture is outmoded. I don't really think it was ever "moded" in the first place. The people who succeed in learning math in a typical lecture-based course are probably capable of learning math without the lecture. After all, most of the learning happens when the student grapples with the ideas for himself after class. Professor Krantz seems to agree with this point.

> To be specific, the old-fashioned paradigm for student learning was that the student would sit in class for an hour and take note; then he would go home and spend three to five hours deciphering the notes, filling in the gaps, and doing the homework.
> *SK, p.16*

Later in the same section, he proceeds.

> The jury is still out on the question of whether students taught with reform methods or students taught with traditional (lecture) methods derive the most from their education. Which students learn more? Which retain more? Which have greater self-esteem? Which have greater interest in the learning process? Which teaching method encourages more students to become math majors? Frankly, we don't know.*SK, p.17*

I leave it to you. Should that jury stay out? Can we do anything to answer those questions? What's that *self-esteem* part about, anyway? Who's talking

[1] I thank Professor Krantz for inviting some of us who have very different views of how to teach math to write short pieces for his book. It's a generous act of a person sincerely interested in the subject of teaching and learning.

about self-esteem? I think most of us agree that the only true self-esteem comes from real accomplishment.

I have heard the charges over and over that people trying to change the way we teach math don't give evidence that what is being done is an improvement over the traditional teaching methods. There *are* studies that indicate improvements in the items listed above. Why are such studies neglected, in particular, here? Steve Krantz should know about at least some of the studies.

Steve Krantz has inadvertently given us some guidance in this direction.

> We make no statement unless we can prove it.
> *SK, p.27*

Of course Professor Krantz was talking about our mathematical craft when he wrote that, and not about the topic of this book. It gives me an opening to complain, though, so I'll take it. We mathematicians seem to leave the tools of our trade behind us when we venture away from mathematics. We tend to abandon clear, supported argument. We all serve on committees, so you know what I'm talking about. I think what Steve Krantz said about the *new methodology* is one such example. There *is* evidence. There's not enough. There's not, as far as I know, a standard set by our learned societies against which we can measure success and failure. Why not? Other groups, including physicists, *have* set such standards against which they measure the effectiveness of different approaches to teaching. We could certainly decide to stop bickering and do the same.

I believe students can learn, but that we can't teach in the way that Steve Krantz believes we can. Lecturing is telling, and telling just doesn't do the job.

> My teacher knows a lot about the subject. He just doesn't know how to tell it to me.
> *A freshman honors student at OSU*

Steve Krantz's *How to Teach Mathematics* is about teachers and lecturing, but not much about teaching or about learning. The student said it all above. The teacher is telling, and the student is assumed to be absorbing, interpreting and internalizing. That may be what a very small percentage of our audience does, and that's probably as it always has been. We mathematicians come from that group. We tend to believe that you either come from that group, or you can't learn and do math. There's a natural tendency to call it the top 10 percent, or something like that. The group I think is really different from the way they were years ago is the other 90 percent. I believe, but can't prove, that the skills people in that majority bring to the task of learning mathematics have dropped significantly since I began teaching in the mid '60s. That's a topic for a different note.

Can we take the other 90 percent and teach them mathematics? I believe we can for a large fraction of that group. If we are to accomplish that, we must become more effective teachers. Effective teaching must include getting students involved and learning. This book essentially ignores students' learning processes. Steve Krantz seems to assume that his audience is full of copies of people who learned the way we did. We were capable of going to class (or not), and then doing the work we had to do to master the material in the text and the assignments just as it was presented. I believe that can happen only when the objects being studied have some meaning for the student, and I believe that most of the students we see in our classes have attached meaning to very few of our objects.

When I started to write this appendix, I went through the Krantz manuscript looking for a statement saying something like, "Lecturing is a good and effective way to teach mathematics." It must be there, but I couldn't find it. Here's one, and it bothers me.

> Turn on your television and watch a self-help program, or a television evangelist, or a get-rich-quick real estate huckster. These people are not using overhead projectors, or computer simulations, or Mathematica. In their own way they are lecturing, and em very effectively. They can convince people to donate money, to change religions, or to join their cause. Of course you calculus lecture should not literally emulate the methods of any of these television personalities. But these people and their methods are living proof that the lecture is not dead, and that the traditional techniques of Aristotelian rhetoric are as effective as ever.
> *SK, p.15*

Steve, whatever is going on in these TV shows isn't teaching. It isn't learning. It's largely emotional response to *convincing* stimuli. These purveyors have mastered the art of appeal to the reptilian brain (see below). It is clearly in the best interest of TV hucksters to keep their audiences from thinking.

The problem I have with the book is that it's about the teacher's teaching, not the students' learning. We know a lot about learning. We should use what we know.

Teaching and Learning

We live in a wonderful time. As we near the end of the 20^{th} century, we have access to whole new fields of evidence about how people learn. We know a lot about how the brain functions, about the structure of the brain, about organization of information, about the differences between information and knowledge, and about the processing of all of that. How can someone write a book about teaching and ignore what we know?

I'll take one simplified model of the brain as a learning machine, explain it briefly, and try to indicate how I believe that model should influence what we do for and with our students. A friendly place to read about this model and its consequences is in the book by Caine and Caine [3]. A more detailed description of the functions described here appears in the recent dissertation of Lee Wayand [13].

The brain's job is to organize itself and the rest of the system it regulates. The human brain consists of three parts; the R-complex (or reptilian brain), the limbic system which contains the hippocampus and amygdala, and the neocortex. That reptilian brain, or lizard brain as Steve Krantz now calls it in private conversation, is basically the brain stem. The next layer is the limbic system and that large surrounding pudding is the neocortex. For purposes of our discussion, the R-complex performs the base level survival functions of the body, from breathing and blood flow, to basic survival skills like fight or flight responses to threat. The hippocampus mediates all sorts of activity, including some information processing and learning. The amygdala is likely a center for mediation of emotion.[2] High level thinking and learning goes on in the neocortex. If you want people to understand and learn, you want them functioning in the neocortex as much as possible. For the neocortex to be in charge, the R-complex and limbic system must cooperate.

There are two basic kinds of learning [1]; facts and understanding are good enough names for my purposes. Facts are what you think they are. They consist of specific instances of features of our surroundings, like specific faces of friends and acquaintances, characteristics of objects we encounter outside ourselves, and, for mathematicians, numbers, formulas and processes. Understanding is the collection of maps and templates we have at our disposal to fit facts into. These templates and maps are the places that facts are connected to gain understanding.

On the next page is a schematic representation of the system needed to get facts into a person's memory.

On the very left we find our senses, and layers of filters one's brain uses to avoid information overload. Most input is ignored.[3] The chamber immediately after the input and filters is short-term memory. It seems to be able to store small numbers (say 7) of pieces of information for only short time. Some of those pieces can be kept alive in short-term memory by forced repetition. Some can be passed on into long term memory by repetition and memorization. Information gets into short term memory and hangs around for a very short time before it either passes through that upper channel to long term memory, or until it is flushed into oblivion through that drain because the brain didn't have any use for it. That flushing is a physical process. It appears that much of the mediation of short-term retention, passage to long term memory, and that flushing is done in the hippocampus.

Take special note of the circles. Those circles around the channel between short and long term memory are sphincters, which can be shut down either by the reptilian brain or the limbic system, and which can prevent the transfer of information either into or out of long term memory. Understanding what activates and what relaxes those sphincters is critical to good teaching.

Information doesn't stay in short-term memory for very long. It seems

[2]Separating emotion from logical thought and learning is impossible [4].

[3]Think about it. Most of the things you hear and see are not important enough to be remembered. They may be important to your understanding your current environment, but they aren't worthy of their own permanent place in your brain.

that it can be kept there longer by repetition, but it won't stay forever in any event. There seem to be two ways things pass from short-term memory into long-term memory. The first and commonest way we build our "facts" database is through repetition or practice. Building this database can be linked to external motivation. Storing facts without understanding almost certainly explains how students can successfully cram for tests and then immediately lose the ability to use those facts again after the test is over. Recall is more difficult. Even if facts reside in the database that is your brain, your internal search mechanisms must have ways of finding them when they are needed. If they aren't attached to objects that carry some meaning (understanding), they may be lost forever. Another way information can pass from short-term memory to long term memory is that it finds an existing hook to attach to. That can be a place that says, "This is like something else I know," for example. Frequently it means that there are slots in various templates for the new information. Of course that means that it is like something else that's in there, because otherwise the templates wouldn't have such slots.

How do those maps and templates work? Here's a simple example.

Imagine walking into a strange airport here in the United States. You see counters at which agents are selling tickets and checking people in, and you see signs pointing you to various gates. You don't have to be told which is which, and you instantly are looking only to see the specific differences that might be important to you. You look for the names of airlines, monitors with flight information, and the like. You don't have to be told what each part of the terminal is about. You know. You can also fill in specific information quickly and accurately, and keep it working for you as long as you need it. No one had to tell you where things were, their function, or the reason for their existence.

This is an example of how templates are frequently spatially organized. Spatial organization can work against our data search as well. To see that, think of running into an acquaintance from work at a clearly independent place, like a beach or mall. You will frequently know you know the person, but won't be able to attach simple information such as a name, because that person is simply out of place.

Our hooks and templates aren't all spatially grounded, of course. Early in the development of the species, they probably were because of our more intimate relationship with our surroundings. By now we have learned to build templates for conceptual structures. We mathematicians have developed elaborate systems of such templates. (In fact, we frequently give them a pseudo-spatial meaning, don't we?) These templates are at the center of concept understanding. Good templates make addition of new facts easier, and make retrieval of relevant knowledge possible. Imagine your own 'function' templates. When, for example, you encounter an expression like $au^2 + bu + c$, and when your hippocampus asks for templates that might have facts about the expression, several things might happen. You would very likely recognize the pattern, and not the actual names of the constants and variables. You would likely think of it as a generic function whose variables might be real or complex numbers or quaternions or matrices or even other functions. You might associate a graph of some sort with it. You would probably not think of this expression as just something to stick into the quadratic formula. (That is what the majority of undergrads do with such an expression once they manage to translate that u to x.)

Let's get on to the process of teaching and learning. Teaching is simply helping students learn. The first step that must be taken is to get the students involved in the process. They are ready to learn, and their minds have learning as their primary occupation. It seems that this makes several jobs for us teachers. One is to keep those channels open between short and long term memory so facts can flow in and knowledge can flow out. A second is to help organize that database of facts into meaningful pieces of knowledge. A third is to help students find the relevant maps and templates that might help them learn and solve their problems. When the templates and maps don't exist, we must help the students build the new ones they need. That comes most frequently from modifying and expanding existing ones.

Those of us who are interested in reform are frequently charged with trying to make learning math simple, or pleasant, or more in tune with students' life styles and experiences. I don't think that's true. The things students do that annoy us as we try to help them learn are probably emanating from inside their reptilian brains. That's certainly true about their hormonal activities, fear and apprehension of math classes, love of MTV, and quest for beer. Those activities may drive a large part of their lives, but I certainly can't get into their learning systems in the neocortex by pandering to the primitive being. If I try to take advantage of, say, students' intimate involvement with MTV or computer gaming, I'm going to be conversing with the wrong part of the brain and real learning won't take place.

Finding the places to connect ideas and facts in students' minds has been recognized as a primary role of the teacher for at least the 2500 years since Socrates. In our lifetime, and our discipline, we have the legacy of George Polya [11] to remind us that our job in teaching as well as in research is precisely that of finding, building and using the appropriate templates. In slightly more recent time, we have the constructivist approach to learning with its primary proponent, Piaget. These days we have the ongoing work of E. Dubinsky and others following the serious ideas of Piaget into real experimentation and research in teaching mathematics.

Many of us believe that the primary tool for the teacher for keeping the channels open, and for helping students find connections in their current knowledge is Socratic dialog. What do we mean by a Socratic dialog? Simply put, the teacher doesn't tell, the teacher guides. The student usually initiates the dialog by asking for help on a problem. The teacher then asks questions about the problem at hand. "What was the original problem? Is this like something you've seen before? What have you done so far, and how did you decide to do that?" The questioning needs to be non-threatening so those channels stay open. The teacher really needs to listen carefully, because in the majority of cases, the difficulty the student has is not expressed in the first few questions and responses. For example, in setting up equations for a simple polynomial interpolation, the difficult part may go back to the student's not understanding that x^2 is not the most general quadratic expression. Once the student decides that the generic quadratic is $ax^2 + bx + c$, a major part of the battle is won. The job of the teacher is to get the student to that point.

Many communication skills are used in Socratic encounters. They are relatively simple to understand, but difficult to use because we aren't used to using them. One example of such a skill is timing. Just as in good jazz music, the spaces and quiet times are frequently more important than the notes. There is a whole craft built up around effective communication of this kind. My familiarity was derived from the community of people involved in what we now call conflict management. The skills for teachers are very similar to those required of a dispute resolution mediator.

Students react in interesting ways to attempts at Socratic dialog. The following is typical.

> He doesn't just answer our questions. He asks more questions until
> I begin to see where my real questions are.
> *Anonymous OSU Student*

Here's my recent favorite quotation from a student who was a participant in this sort of discussion for several months. It represents an extreme view from a student's perspective of what's happening during such encounters.

> They have this thing. You ask them a question and they just stare at you, and suddenly you know the answer. It's really annoying. (Now they are trying to teach us how to stare at students over the Internet.)
> *An OSU Student and Distance Mentor*

I believe that Socratic dialog between students or groups of students and instructors is where learning begins. I believe that very few students can sit and assimilate facts and turn descriptions into knowledge and processes if I mostly just stand in front of them announcing new ideas. I'm sure they can't do it *during* a lecture, so in Calculus&*Mathematica* courses[4] [5], we present students with new ideas in Mathematica notebooks. They are encouraged to explore the ideas by experimenting, discuss their ideas about what is happening, and build their own templates for understanding. Obviously students can't be expected to invent all of the ideas.[5] What they can do is become familiar with processes and ideas before they are formalized by the teacher. Only then will the ideas have some meaning attached (templates built).

Obviously class time is not where the bulk of learning occurs. Our C&*M* students spend long hours working at home and in the lab. In fact, if one is to believe their estimates of time spent on learning math, our students are involved for many more hours than the typical student is. Of course, they aren't usually spending Steve Krantz's recommended 3 to 5 hours outside class for each hour in class. The last time we tried to estimate the time, the average was somewhere between 15 and 17 hours per week per student in the lab for our 5-hour courses. Class time is spent helping students strategize for learning.

To reiterate, I see my role as that of questioner. My job is to help students find the hooks that might accommodate new facts. My job might be as simple as getting the student to admit that $au^2 + bu + c$ is simply an alternate spelling of $ax^2 + bx + c$. I keep asking questions as they search their memories. All the time these discussions are going on, it is my job to keep those learning channels open. I can't tell students how far back into their memories they may need to go to find a hook they need for the task at hand. They must find that for themselves. I can only guide.

At this point, all of this probably sounds like an impossible task. I am asking you to sit with students patiently guiding their thinking back to places in their knowledge that impinge on the problem they are working on. Well, it is difficult. On the other hand, it is probably not as daunting a task as it appears at first. For one thing, you should be talking to small groups of students most

[4]Most of my experience with using the ideas in this note come from my involvement with Calculus&*Mathematica* classes. I'll refer to them throughout the rest of this note.

[5]S.K. (p.104) includes as one of the hallmarks of the reform movement, in a bulleted list, "Students should discover mathematical facts for themselves." I'll wager that Steve Krantz and I read completely different meaning into this sentence. I'll bet Thomas Kuhn [9] would say that Professor Krantz and I are encountering a paradigm shift.

of the time rather than individual students. For another, you'll soon find that the paths to the correct hooks and templates are very similar for the majority of students. You'll also find that giving students related tasks to perform can shorten the process. Finally, the real experts at helping students find the way are other students. Once a path has been found, students are happy to share what they have found. Surprisingly enough, once they are used to the Socratic business, they also don't simply tell other students what they have discovered. They try to help their peers find their own way.

So what about lectures in my courses? At the end of each Calculus&-*Mathematica* lesson, as the students try to summarize and make sense of what they encountered, we have a review session over the ideas the students wrestled with as they worked. This is the only time we do anything like a lecture in the class.[6] I don't start new lessons by telling the students what I want them to do.

This is difficult stuff for mathematicians to cope with. Occasionally, you must suspend disbelief as you listen to the places students can take you in their quest for hooks. Here's an example. Early in my differential equations course, I ask students to check on whether or not various functions are solutions to differential equations and some initial value problems. They have trouble, probably because checking that one function equals another function isn't something they've done much of. In order to clarify what I'm asking, I might start a discussion with, "Check and tell me if $x = 7$ is a solution to $x^2 + 6x - 7 = 0$." Most of the time, a sizable number of students in the classroom will say, "No. The solutions to that are $x = -7$ and $x = 1$." When I ask what they did, they tell me that they dug out the quadratic formula, plugged in the numbers, and got those answers. When I ask if anyone did it differently, I rarely get anyone saying they just plugged 7 in and didn't get 0. Why is it worrisome? It indicates to me that they have only programmed responses to much of the mathematics they encounter. The processes they bring usually don't come with any understanding; they are rote procedures. We have trained them to react rather than analyze. The programmed response to a quadratic expression is plugging something into the quadratic formula. I change the problem to something they don't know how to solve, like $x^4 - 28x - 701 = 0$. It takes rather a long time to get them to plug the 7 in for x and see what they get. Then we get back to the problem of whether or not $y(x) = sin(2x)$ is a solution of $y'(x) = 2y(x)$. Following a great deal more discussion, they frequently say that it happens at lots of points, and give me as evidence a graph of $2cos(2x)$ plotted together with $2sin(2x)$. They point to the crossings, and say, "Here." It turns out that we are still a long way from having them internalize the notion of functions being equal, and checking that some expression is a solution to a given differential equation.

One more time, here's the challenge. If it's true that students can only build new knowledge upon their existing knowledge, we should know what that knowledge base is. George Polya [11] opened that can of worms for us math-

[6]There is a frequently encountered exception to that. If I spot something that needs a quick redirection for several groups of students, we'll take a short trip to the board for a discussion of the particular sticking point.

ematicians a long time ago. What do our students know? What does know mean? What is knowledge? A group of my students announced in a discussion we had this spring that knowledge is information that's usable. Postmodern philosophers have made knowledge into a deep and complicated activity. At virtually any level, though, we need to know what usable knowledge our students have at their disposal. That, to me, doesn't mean, "What formulas do they remember?" or, "What formulas don't they remember?" I don't know how to answer the question about what students know and don't know. I know even less about how I might assess what they are capable of doing with what they remember. Look at the previous paragraph. What are the students struggling with? Is something missing in their 'facts' database, or so they have rusty templates for interpreting 'equals', or what? The best I can do for now is to question students as the need for help arises, and hope that together we can find a way to help them understand the question.

We mathematicians tend to talk to the top 10 percent of the student body: that part capable of learning mathematics well, and as we learned it. Perhaps we are simply so proud of ourselves that we have failed to notice that other people have skills and abilities that can make their understanding of mathematics as real as our own even though it may look different to us. We have always known that some mathematicians deal best with geometric and spatial objects while others deal most fluently with symbolic representations. Our teaching, though, seems to concentrate most strongly on the latter, and reward for hard work comes to those who do well with the symbols and the formal argument. In other words, success in mathematics continues to be, for the most part, limited to people who think and learn the way we did. Are there other ways? I think so.

If our brains develop according to our life experiences, our environment, and our inherited family traits, it makes sense that each of us has a different cognitive structure to deal with observation and learning. When I was growing up, the preferred measure of intelligence was IQ. Tests were given which measured linguistic and mathematical propensities, and declared intelligence or lack of it on the basis of the results. That seems to be a rather narrow view of intelligence. Don't you agree? What about other 'intelligences'? Going back to Dewey [6] and Jung [8], and moving forward to Gardner [7] and many others[7], we are forced by reason and observation to consider other capacities. Among these are the ones Howard Gardner added to the Verbal, and Logical/Mathematical[8] upon which IQ testing rested: Kinesthetic, Spatial, Interpersonal, Intrapersonal, Musical, and a newer addition, Natural. We should be prepared to take advantage of the different learning styles our students bring to us.

Most of the reform teaching projects expand the approach to teaching and

[7]One would do well to look at the ideas of Myers and McCaulley [10], and Gregorc [12], [2]. A synthesis of many of these ideas is emerging from Bernice McCarthy's 4MAT model (`http://www.excel.com`).

[8]I think this is a terrible name. It certainly doesn't encompass the wide variety of skills that might describe success in mathematical adventures.

learning to accommodate a broader learning style base. The first clear statement of that came from the Harvard consortium's rule of three (now four): Symbolic, Numeric, Graphical and Verbal. Quite simply put, a student should first be able to approach a new mathematical idea through that student's own strengths. After that, the concepts should be viewed through each of the other lenses, and the student should be able to put real flesh on the bones of the concepts by making the connections between the four perspectives. Is anything different from what we have always done in our classes? Probably. For one thing, a student should be able to approach a concept from each of the different points of view. For another, we all expect students to be able to give clear and concise explanations of what they do, observe or calculate orally or in written form. In Calculus&*Mathematica*, we expect such descriptive writing in each problem solution.

Go back to the illustration of the learning system above for a minute. That simplistic cognitive model can also let you see that there are different ways of interpreting and processing information and experience. One interpretation, from the work of Gregorc, suggests that one perceives new information best from either a concrete or abstract presentation, and then processes that information internally either sequentially or randomly. One can simply read the descriptions of what these terms mean and make good guesses about who will succeed and who will fail in our traditional classrooms. We can accommodate more people if we simply make our teaching more flexible than it currently is.

There's one more book that's a bit off the beaten path that I recommend to you now. It's *In The Mind's Eye* by Thomas West [14]. The premise is straightforward. Too little attention has been paid in the past to the visual learner. The contention is that we live in a visual age, and that there are many people who are labeled as being dyslexic, or learning disabled, whose ability to learn in standard modes is less than what our educational system expects.[9] These people generally have very strong spatial abilities. According to Tom West, we are also entering a far more visual age. It may be that the pendulum will swing, and that the greater value may soon be placed on people who excel in this mode of information processing.

There's not room to do any of the ideas introduced above a fair hearing here. Why don't you talk to some people, visit a bookstore and read some more about all of this?

Epilog

Those of us working toward change are frequently accused of trying to simply make math fun, or make math easy. I don't agree, of course, with that assessment. I expect my students to work very hard, and I expect a lot from them. One C&*M* instructor at Ohio State this past year decided that the text and electronic lessons were deficient, that they didn't make the students master enough facts. He proceeded to fix all of that by taking the students from the lab for

[9] Here I go again. This is yet another topic for discussion.

regular sessions on the topics left out of the course. He made his expectations very clear, just as we all usually do in a traditional classroom. He expected the students to be able to perform a clearly defined list of manipulative exercises, the kind one finds on most calculus exams at our colleges and universities. We had a protracted conversation in e-mail. Here's one thing he said.

> I can show you my student teacher evaluation forms if you'd like which are in general very positive and as a matter of fact there's one which I'm looking at right now which explicitly thanks me for taking the students out of the classroom once a week to explain/review the material.
> *Calculus Instructor*

I'm not surprised, are you? This instructor gave the students clearly defined tasks of the sort they had experienced for their previous twelve or thirteen years of math. He took them to a place where he could explain clearly how they should answer each of those questions, and then tested them on what he showed them. What could be easier for a student? A student has a success template laid out in front of her. She works hard at practicing the skills the instructor recommends, and if she does that well, she'll succeed. There's no need to impose any internal meaning on the processes, and there's no reason to think that much, if any, of the material will ever come back to haunt her. This student can get a good grade in this course without having anything she learned be transferable even to the next course.

Ask yourselves again, "Who is trying to make it easy for the student to succeed?"

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Reflections on Krantz's *How to Teach Mathematics* A Different View

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1. Introduction

When Steve Krantz asked me to write, for the appendix of this book, an essay expressing my views of teaching, I thought it would be useful to do this in relation to the book itself, rather than just give my views in isolation. So my first step was to figure out what the book was all about. I began, naturally enough, by reading it. And I discussed a lot of things with Steve. And I thought about it. Here is what I think the book is.

Krantz has produced a serious reflection on a traditional approach to teaching and learning. He acknowledges the existence of views other than his, but although the book goes far beyond a mere reference to these alternatives, it is considerably less than a balanced presentation of a variety of different approaches to collegiate mathematics education. Even with these appendix essays, the book retains its strong reflection of the very personal views of the author, based mainly on his many years of experience in the classroom together with some, not overly extensive, awareness of past, present, and future investigations into the teaching and learning enterprise.

We get two things from what Krantz has written and both are welcome. First, the text, together with the appendices (some of which I have seen as of this writing) contribute to moving our considerations of teaching and learning collegiate mathematics from an argument in which we are all fighting to defend our positions and attack those with whom we disagree, to a set of reasoned discussions in which we clarify points of agreement and disagreement, provide justifications for both, and ultimately search together for the syntheses that alone can make significant improvement in the breadth and depth of what mathematics our students learn.

Second, we learn from this book that although there really are identifiable categories of viewpoints such as traditionalists and reformers, it is not the case that a member of a category differs in all of her or his ideas from any member of a different category, nor do all members of a category agree with each other on every issue. What this realization helps us with is the need to focus on ideas and not on those who hold them. When we do this, we find many points of unexpected agreements and disagreements varying with the issue and the individual. I find that the most important precondition for me, in the words of Secretary Riley, to stop fighting and get on with the job of improving education, is to understand that Andrews, Askey, Hughes Hallett, Krantz, Uhl and everyone else might, on any given issue, agree or disagree with me.

So, in the spirit of this latter point and in hopes of contributing to the former, I will try in this essay to reflect on a number of issues that arose for me in reading Krantz's book. My goal is, in each case, to explain and perhaps justify my position, to express my agreement or clarify and possibly justify my disagreement. The topics I will consider are: beliefs and theories about the nature of learning; teaching methodologies; the use of technology in mathematics education; applications and pedagogy; and some specific pedagogical issues.

2. Beliefs and Theories about the Nature of Learning

In this section I would like to talk about traditional vs. reform approaches to mathematics education, the particular approach called constructivism and its relation to discovery learning, the relation between knowing how to calculate and understanding concepts, the intrinsic difficulty of undergraduate mathematics topics, and how we can try to decide what are the effects of a particular pedagogical method.

2.1. You @!∗#?! ∗ ∗ Traditionalist!

Krantz believes that mathematical knowledge is something, in some sense or other, very definite and concrete, that exists in the minds of some people (usually called mathematicians) and that it is possible to take this entity and transfer it to the minds of other people (usually called students) using media like speaking, or writing, or showing pictures. I believe, on the other hand, that mathematical knowledge is much more elusive, being a characteristic of the behavior and thought, external and internal, active and reflective, of *all* people, in varying degrees of quality and sophistication, and that it is built by individuals, acting in social contexts as they try to make sense of certain situations in which they find themselves.

Something like Krantz's beliefs have been held by a large number of people for a very long time. My position is relatively new and although it has a name that is becoming, unfortunately, a buzzword (constructivism — there I said it, I even carry a card!), it is not a position really held by very many people. It is in this sense that I say Krantz is a traditionalist and I am a reformist. It only means that he is trying to hold on to what is best about a going concern, and I am trying to break some shells and make a new omelet.

Now you might think it the epitome of arrogance for *me* to say what *Krantz* thinks. Indeed, doing this contradicts one of my most deeply held beliefs (e.g., that one person can never really know what is in the mind of another person). But I only put it that way to catch your attention. What I really mean is that what I just wrote is one way that one might interpret many statements, such as the following, by Krantz in this book (the comments in square brackets are mine.)

> "...you [the teacher] must be conscious...of what are the key ideas you are trying to communicate to the students..."
>
> "If you [several bad things some teachers do sometimes] then you will not successfully convey the information."
>
> "...the teacher who [several good things some teachers do sometimes] conducts a good class."

"Remember that you are delivering a product."

If we were to schematize the situation suggested by such comments, it is that of an active teacher trying to do something to passive students.

2.2. I Admit It, I'm a Constructivist!

I am convinced that learning does not happen in the way suggested by Krantz's comments. I believe that a person learns by actively trying to do something, or make sense of something and must, almost consciously, make fundamental changes in the make-up of that vague entity called her or his mind. It is making mental changes that I call constructing and it is my belief that the teacher can only act indirectly to get the students to construct these changes. That makes me a constructivist in my approach to mathematics education.

So what does a constructivist do by way of research in teaching and learning together with curriculum development? There are many people engaged in this enterprise and they have many different answers to these two questions. One pair of answers that a number of people have adopted in their work is this. Research means theoretical and empirical studies to understand the nature of the specific mental constructions that a person might make in order to understand a particular mathematical concept. These studies should point to pedagogical strategies that might help students make the mental constructions that are proposed as a result of this research. Then, of course, curriculum development involves the design and implementation of courses in which this pedagogy takes place. The instructional treatments should focus on getting students to make the specific constructions and it should also, guided by what we can learn from research, make use of innovative pedagogical approaches such as cooperative learning and writing computer programs as a way of making mental constructions.

Consider, for example, mathematical induction. In some research that was done over a decade ago, an important difference between students who were successful and those who were not with making proofs by mathematical induction appeared to emerge. It seemed that the successful students were constructing a mental process — or function — that took a positive integer and returned a value of true or false depending on some operation. For instance, if the problem was to show that for n sufficiently large, a casino with only \$3 and \$5 chips could accommodate any number n of dollars, then successful students seemed to be thinking of a function that acted on a positive integer n and returned the truth value of the proposition: There exist positive integers k, j such that $n = 3k + 5j$.

For many students this process conception represented serious progress from being restricted to thinking about a function in terms of plugging numbers into algebraic expression, which we would call an action. It was even harder for students to be conscious of an operation *on a function* which converted such a boolean valued function P of the positive integers into another such, say Q, in which $Q(n)$ is the truth value of the implication $P(n) \implies P(n+1)$. This requires thinking of a function as an object to which higher level actions can be performed.

A somewhat more detailed and extensive version of these comments came out of a research study that combined theoretical analysis with empirical results to obtain what was called a "genetic decomposition" or "cognitive model" of the concept of mathematical induction. The next stage of the study was to develop and implement an instructional treatment that helps students make such constructions. For example, to construct the above function P, we had them write and use the following computer program.

```
P := func(n);
        return exists k,j in [1..n] | n = 3*k + 5*j;
     end;
```

For converting the implication we first had them build a "machine" for doing this in general:

```
convert := func(F);
              return func(n);
                        return F(n) impl F(n+1);
                     end;
           end;
```

And then they could just do

```
Q := convert(P);
```

This code is written in the computer language **ISETL**, which is what we call a *mathematical programming language*. Its syntax is very similar to standard mathematical notation and it can express almost every finite mathematical construct (such as quantifications, functions as sets of ordered pairs, sets and subsets defined by conditions, finite sequences, etc.) in the language of mathematics. The idea is that by making mathematical constructions on the computer the student tends to make corresponding constructions in her or his mind. See [5] for a discussion of **ISETL** and its use in mathematics education.

The study found that using such an idea — together with some other pedagogical strategies, students developed what appeared to a surprisingly deep understanding of making proofs by mathematical induction — and they could make such proofs, even when the problems were somewhat different from what they had been practicing on!

You can see the details of this study in [3], [4].

This kind of approach has been applied to a fairly large number of concepts in undergraduate mathematics and courses (including textbooks, lesson plans, sample exams, etc.) have been developed in discrete mathematics, precalculus, calculus and abstract algebra. It is a general research program that I, together with several others, are trying to implement. An overall description of the research we are doing can be found in [2] and you can find links to specific published research reports and textbooks at the following WEB addresses:

http://rumec.cs.gsu.edu/

http://www.cs.gsu.edu/ matjbkx/edd/mypapers.html

Finally, since I am talking about constructivism, I think I need to say something about discovery learning. There seems to be some confusion here about what is being advocated. Krantz says that the "reform school of thought" favors discovery and that one of the "hallmarks of the new methodology" is that students should discover mathematical facts for themselves."

Not exactly. One does not have to be an educational traditionalist to realize that what took hundreds, if not thousands of years for mathematicians to discover is not likely to be figured out by very many students — even if they do have lots of giant shoulders to stand on, powerful computers to work for them, and intelligent, sensitive teachers (all up to speed on existing research) to guide them. It is almost insulting to hear so many people suggest that reformers are so misguided as to not realize this. If we set before our students mathematical task to perform *before* they have learned the mathematics then they are not likely to invent a proof of the fundamental theorem of calculus.

So why do we do it? Why do we try to get students to work on mathematical problem situations that use mathematical concepts and methods they have not yet learned. It is not that, given the appropriate situations, they *are likely* to realize on their own that there is some kind of inverse relationship between velocity and area, between derivatives and limits of Riemann sums; or that they *can* (and I can tell you that many do) invent the chain rule after reflecting on some at first mysterious examples. All of these things happen in many reform courses and they happen with relatively small loss of "coverage". But they are not the most important reason for giving students an opportunity to discover various things in mathematics.

The real reason is the word I just used—opportunity—and it is an idea that goes back to Piaget. If you give your students the *opportunity* to discover some bit of mathematics, then whether they succeed or not, it seems to be a psychological fact that much of the attention and mental energy they put into figuring something out is directly transferred to understanding the explanation when it is given by a member of the student's group, or another group, or the teacher. And since you are not planning on waiting until everyone (anyone?) has made the discovery, you can cut it off and move on whenever you decide, thereby controlling the amount of time spent on *trying* to discover something.

2.3. A Mathematician? You Must be Able to Add up Really Long Lists of Numbers.

We all know what happens at cocktail parties when we own up to being mathematicians. People think that all we do is arithmetic operations with numbers—lots of numbers and big ones, too. Traditionalists accuse reformers of ignoring calculations and reformers confuse traditionalists with mindless guests at a party. Both are wrong.

On one level, I am somewhat bemused that there is an argument here. How can anyone in the mathematics enterprise, from high school teacher, to industrial user, to researcher, even conceive of mathematics without lots of heavy computations? On the other hand, how can anyone who actually understands mathematics think that calculation is the whole story. To be honest, I don't think I know any traditionalists or reformers who hold to either of these views. I think we all believe that understanding mathematics means that you are able to perform calculations and that you have some understanding of what those calculations are about.

You don't have to be a whiz at calculations, but you do have to be able to do them, however slowly and however many times you have to repeat them to avoid errors. Technological tools can and should be used—but as aids, not to replace the ability of the person who uses them. To be specific, you have to *be able* to take the derivative of any composition of elementary functions, even though you may often choose to use a computer algebra system to do the job, or to check your work.

As to understanding, you have to have the proverbial "feel" for calculations so as to have a rough idea of how they are going to come out without necessarily doing them, you have to be conscious of patterns, you have to know what the calculations are useful for and how they relate to other situations, mathematical and otherwise, and you have to know *why* particular calculations are made in particular ways.

The fact is that both calculations and conceptual understanding are essential parts of mathematics and they are completely dependent on each other. Differentiation is an incredibly useful tool for solving a multitude of problems from the everyday world. It is also an operator on certain function algebras that is linear but not exactly multiplicative—and it transforms composition to multiplication. I think it is important to understand these abstract properties of differentiation, but I can't imagine a very deep version of such understanding that is not based on a thorough knowledge of the rules for calculating derivatives.

Surely everyone agrees with what I have just written. So why is there a controversy? The controversy comes because there is a very large number of students who have difficulty with either calculations or conceptual understanding or both. Our differences here are about how to deal with this. I think it was a wonderful idea that maybe, if you could use a computer to do the calculations, the student would be freed to spend more time and energy understanding the concepts. Either that would be enough or it might turn out that going back later, the student could better learn how to calculate. So maybe we should stop insisting the students learn the calculations, let them use the technology and we can get on with the concepts

Unfortunately, it seems fairly clear to me that this does not work. Krantz is right when he points out that "...there is no evidence ...that a person who is unable to use the quadratic formula" by hand but can do so with a machine will "...be able instead to analyze conceptual problems". My experience suggests that calculating derivatives is hard for many students but understanding differentiation as an operator on functions is much harder. There is simply no way around the pedagogical problem that we must find way to get our students to be better at both calculations and conceptual understanding.

I think, incidentally, there are ways to use technology to help do this and there are several references given by Krantz (for example, the paper by Heid and my own, joint, work in calculus with several others.)

An important point for teachers who plan to think about their students' work with calculations and conceptual understanding is my observation that the strongest resistance to de-emphasizing calculations can come from the students who (with considerable justification) feel extremely insecure when asked to reflect on calculations they can do by hand only haltingly. In the context of a theory I work with, there has to be a strong base ability to perform actions (calculations) before an individual is ready to move on to thinking about processes (reflection on the actions).

2.4. Yes, Barbie, Math is Tough!

I think Krantz is wrong when he says: "There is no topic in the [calculus] course that is intrinsically difficult. We merely need to train our students to do it." Aside from the fact that his second sentence here is more of a focus on actions as opposed to reflection than I think wise, I am not sure what could possibly be meant by the term "intrinsically difficult". Given any topic (in calculus or any mathematics subject), any students, and any point in their development, there will be some for whom the topic is hard and some for whom it is easy. Both my experience and my research suggests very strongly that the idea of a function as an object, so that the derivative of a function is a function, or the solution to a problem can be a function, is hard. It was hard for entire societies and required a long historical development. I think that almost everybody has to make some serious mental changes to move from thinking of a function as something that does something (process) to thinking of it as something to which something is done (action on an object). Whatever the phrase means, I think that developing the ability to understand a function as an object is "intrinsically difficult".

The point is important because we really have to decide, regarding such examples, whether we can get by with pedagogy focused on just training our students to do it, or if we have to develop substantial pedagogical strategies to help our students overcome a major obstacle.

2.5. So Who's Right?

I mean who is right in all controversies about how people learn and what teachers can do to help? I think Krantz's answer to this question expresses the situation exquisitely in four words: "Frankly, we don't know".

But we had better find out! I think that research is one way to do so, but we cannot expect the kind of research that makes a small number of studies (or even a single one) and allows us to say that this or that pedagogical approach is best—or even better. Our domain is much too difficult, compared for example, with medical research and even there we have extensive studies over long periods of time that tell us things like thalidomide is relatively harmless and watching atomic explosions can be done with impunity.

The kind of research we need is basic, long term and must relate to theoretical analyses in addition to the gathering of data. Research can begin to tell us a few things, such as that it helps students understand and succeed with proofs by induction if they are able to think in terms of functions whose domains are the set of positive integers and whose ranges are the set of two boolean values. But we need to integrate this research gradually and intelligently with our teaching and this is one reason I hope that the fledgling field of research in undergraduate mathematics education becomes accepted as one of the mathematical sciences.

But research will not be enough. We need discussions in which people, no matter how much they disagree, are working together to find solutions to our educational problems and not just score points off each other. We need working together in modes such as this book and its appendices. Indeed, one of the most important contributions of Krantz's book is those four words above, together with the clear thrust of doing something to find out. Perhaps here is a place to point out, as I have done elsewhere, that Piaget (who is a great role model for me) habitually dealt with major figures who disagreed with him by inviting them for a year or so to Geneva to work together with him on a project—not to decide who is right, but to synthesize their opposing views (the book by Beth and Piaget listed in Krantz's bibliography is one product of this custom). As far as I can tell, in the field of collegiate mathematics education, the only person who has really done anything that even moves in that direction is Krantz in writing this book, in inviting these appendices, and discussing with many people what are their objections to what he is writing.

3. Teaching Methodologies

Throw out the old and bring in the new! Well, perhaps, but let's be careful. I would like to talk about why I don't think the lecture method is a very good idea. I would also like to describe some possible alternatives. This brings up the question of how to decide what kind if teaching one should do? Having made that decision, it turns out that the work is not over, it has just begun. Learning how to use a particular teaching methodology is not automatic. We all went through many years (decades for some of us!) being subjected to and using what very roughly might be called a traditional pedagogy. So we will be prepared to use that if it is our choice. I hope it won't be. But that means that there is a lot of work to be done in developing the ability to use a pedagogical strategy on your students that is different from what you experienced.

3.1. Do Lectures Work?

Krantz says that he is not ready to give up on lectures because:

> "They have worked for thousands of years, in many different societies, and in many different contexts. And they have worked for me."

Is this really true? Forget about "working" for a moment, and let me ask if lecturing as we know it has been happening for "thousands of years". Did Socrates or Plato lecture? Is this what the mathematical monks of the Middle Ages were doing? What was Fermat's classroom presence like? Was Newton effective in his presentations to students? How large a class size did Galois have to deal with? Was lecturing the way European students of the 18^{th} and 19^{th} learned mathematics? Or maybe Krantz is referring to thousands of years of lectures on mathematics in China?

I am not an historian and I don't know. From what little I have heard, however, I think that the practice of lecturing as the main teaching methodology for university mathematics is much more recent. I suspect that instead of "thousands of years", we may be talking about hundreds of years, or perhaps even decades. Whatever is the extent to which lectures have worked, and whatever is meant here by "work", I am not so sure there is an extensive history supporting this particular strategy.

What cannot be doubted, of course, is Krantz's claim that they have worked for him. Well, something worked for Krantz and the multitude of other successful research mathematicians we have produced, let us say in this century. It is possible that for these particular individuals, just about anything would have worked (even cooperative learning!) Even they did learn mathematics by attending lectures, this collection of individuals is hardly typical and certainly not very representative of the student body we are working with in our society's great experiment with mass education at the post-secondary level.

What does seem clear now and for this student body is that lecturing is not working. This is attested to by reports of minuscule attendance at the lectures, poor performance in tests based on those lectures, dissatisfaction on the part of teachers of these students in subsequent courses (in mathematics and courses for which mathematics is a prerequisite) and what seems to be a decline in the rate of successful completion of mathematics courses.

If we can all agree that this problem exists, there are certainly very different views about its causes, and what to do about it. Krantz takes the position that lecturing would be sufficiently effective if we were better at it. I differ with that on purely personal grounds. I think I am an excellent lecturer—but I don't think my students got nearly as much out of my lecture courses as they do today in my courses which use other pedagogies.

Others argue that it is the students' fault. The background they obtained in high school (where they were taught by teachers that are largely products of lecture-pedagogy) is too weak, or their attitudes are all wrong, or the conditions under which we teach are inadequate. Be all that as it may, if we have a huge enterprise (collegiate mathematics education) which is not working, then it is pretty unlikely that all of the fault lies in places other than the methods teachers use in that enterprise. I think that as mathematicians we are responsible for working to make changes in the overall system to help deal with our problems, but that is a long-term operation. In the meantime, we must figure out how to do the best we can with the students we have and the conditions under which we work. In my view, that means we must look for alternatives to the lecture method.

3.2. What Are the Alternatives to Traditional Pedagogical Methods?

There is a multitude of new pedagogical strategies in collegiate mathematics education that many people have been developing and implementing over the last decade. They include various ways in which technology can be used, cooperative learning, replacing some lecturing with methods in which students are more active, writing, and the use of history.

It would be very nice if I could point to a place where all of these methods are described. Now is the wrong time for this. We are too much in a period of new ideas, revising first attempts, discarding some approaches and pursuing others that seem more promising. Krantz gives some information on what is available and MAA OnLine is another source. But I am afraid that the interested faculty member must do a lot of digging in the library, read publications of the MAA and AMS, attend sectional and national meetings where some of these approaches are discussed, and generally be on the lookout for new ideas as they emerge and are reported.

There is some discussion above in my section on constructivism where I talk about some of the pedagogical approaches I have been working with. This is in the context of pedagogy that supports a constructivist view of how learning takes place.

At some point in the future, we will have to think about producing compendia of new pedagogies. Amongst other issues this will raise is the question of effectiveness. What do we have to say to the working teacher about the relative effectiveness of these new methods?

3.3. How Does a Teacher Decide on What Pedagogical Approach to Use?

There is not a lot that can be said about this today. As I commented earlier in agreement with Krantz, we simply do not know how to decide with any great degree of certainty on the effectiveness of a particular teaching method. It is very much like parenting. There are lots of views and many things to read, but in the end, each teacher, like each parent, must decide as best he or she can what pedagogical approaches to adopt.

I must, however, insert a word of caution here. Some people might interpret the previous paragraph as stating that, for a given teacher, whatever "works" for her or him is the method that should be adopted. The danger here is in restricting the concern to the teacher. As Krantz puts it, "...we should each choose those methods that work for us and for our students."

This is not so easy as it sounds. Aside from the difficulty in deciding what works for our students, we must acknowledge that sometimes, what works best for the teacher, may not work very well for the students. To put it extremely, the teacher who faces the blackboard speaks in a monotone, writes out the text on the board, works a few illustrative exercises, and assigns homework, may be using a methodology that "works best" for her or him (in the sense, for example, of minimizing the distractions from other issues in the teacher's life). But I hope we can all agree that this is not likely to be an approach that works very well, much less best for the students involved. At the very least, it is not at all clear that a teacher alone can always determine how effective is her or his approach to teaching.

3.4. What about Implementing a New Pedagogy?

Here, I agree with Krantz that, as hard as it is to decide to implement a new pedagogical approach, this is only the beginning. As Finkel and Monk [7] point out, it is very difficult for a mathematician, with no background in educational methodology, only the experiences he or she had as a student, and in some cases, long years of practice with a method he or she has decided to replace, to actually change the way he or she teaches.

First, one has to make the decision. Then you need to learn about the method you have chosen to implement—how it works, what you can use from previous methods, what changes you need to make. For most people, this is still not enough. You need some kind of mentored experience in actually using the new methodology.

There are some indications that mathematics departments are beginning to introduce courses in pedagogical methods for graduate students who will have a career in college teaching. There are also a number of mini-courses and workshops organized by the professional societies. Krantz has mentioned MAA's Project CLUME and, in fact, MAA has a program of Professional Development. Finally, those of us working in educational reform produce a great deal of written material that can be helpful and Krantz has referred to these.

All of this is a good start, but I am afraid it is still not enough to make systemic change in teaching and learning collegiate mathematics. I hope that readers of this book will not only pursue the opportunities for professional development that do exist, but also will push for an extension of these activities to a sufficient level.

4. The Use of Technology in Mathematics Education

The use (or not) of technology is one of the most controversial issues in mathematics education today. We appear to be totally polarized—there are those for and those against. To me, this is completely ridiculous because there are many ways in which technology can be used in mathematics education and I differ with some people who advocate some of those ways at least as much as I disagree with those who are more or less against any significant use of computers.

For example, both Krantz and I deprecate what is called programmed learning, or Interactive Tutoring Systems. Krantz worries, correctly in my view, that some of these systems may not provide enough opportunity for the student to ask questions and the teacher to respond. I am also concerned about the ways in which these systems try to get students to think about concepts. I do not believe that mathematics can be reduced to a collection of goals and subgoals together with not very rich ways of connecting them.

I also have a lot of concerns about the use of today's sophisticated calculators. In earlier times, I did not hear students say that the limit of a function at a point is the value of that function at a nearby point. Is this new misconception due to certain ways in which calculators tend to be used? I have similar troubles with the graphing capabilities of hand-held calculators because I think they can focus the students' attention on the least interesting examples.

But this particular issue will soon go away. Hand-held computers are rapidly approaching the functionality of desk-top computers so that soon we will have to look for something else to fight about.

So we have to talk about the ways in which technology can be used and I also want to explain the way I think is the most effective and why.

4.1. What are the Different Ways in which Technology Can be Used to Help Students Learn Mathematics?

I have written elsewhere [6] about the different ways that I think technology can be effective. They are, roughly: using graphics capabilities to *show* mathematical phenomena; using the computational abilities of a computer algebra system to *do* mathematics; and using the expressibility of a programming language to *construct* mathematical entities on the computer.

4.1.1. Using Technology to Show Mathematics

There is no question that we can produce incredibly wonderful pictures on a screen using today's technology. On the one hand, I think that makes many mathematical situations more real and accessible to students—at least as phenomena to be explained, manipulated, and perhaps understood. This can be very helpful but, in my view, entering an expression, pressing buttons and looking at a screen, or even manipulating the screen with a mouse is a little too passive in terms of the mathematics involved in producing the picture.

To take a simple example, consider entering an expression that defines a function and having the computer produce a graph. Suppose even that a table of values is produced, and that it is possible to manipulate one or more of these "representations" (expression, picture, table) and have the others change correspondingly. Even in such a sophisticated system, I do not see that the student is helped in any way to understand the connection between the various processes of plugging a number into an expression to get a result, looking down one column of a table for a number to see what is in the corresponding place on the other column, or locating a point on the horizontal axis and seeing how far up you have to move until you hit the curve. Conceptually these are all the same process and it is very important for students to understand that. I am not sure they do. Yes, students can learn from such technological systems that if you add a constant to the expression for the function the graph goes up. But what is it that helps them realize that the reason is that you are still computing the same values to place on the graph, but now every answer you get is increased by the constant, and that means higher up on the vertical axis?

My conclusion from all this is that using technology to show mathematics to students can be helpful, but it will be much more effective if used in conjunction with other activities.

4.1.2. Using a Computer Algebra System to do Mathematics

Using the power of **Maple** or **Mathematica**, students can learn to perform highly sophisticated and powerful algorithms. The idea is that making use of mathematics in this way is going to help the students learn elementary versions of the mathematics involved. Perhaps, but I am not yet convinced that using a computer algebra system to do applications that involve Padé approximations or Tchebycheff polynomials is going to help the first year calculus student understand how to compute the McLaurin series expansion of $\sin x$ and the issues that arise when you reflect on what relationship that series has to the sin function. It is possible, but there is absolutely no evidence in favor of this approach—even less so than evidence for other claims, both traditional and reform.

Some people take the very opposite view of using a computer algebra system to reduce the time and effort spent on learning standard manipulations. Please pay very careful attention to the fact that I said "reduce" and not eliminate. My remarks earlier about the importance of calculations still stand. What I am saying is that we can use a computer algebra system to do as well or better than in traditional classes—in less time. Here there is some research here for example, the study by Kathy Heid [8].

Yet another approach is to use a computer algebra system to provide data on the basis of which students can discover mathematical relationships. For example, before talking at all about the rules for differentiating combinations of functions, I ask students to use **Maple** to calculate about a page (closely spaced, I admit) of derivatives of elementary functions using **Maple**. Then, in class discussions, I can get most students to invent the rules for derivatives of sum, difference, scalar multiple; many will invent the product rule; and a few will come up with the chain rule.

Of course all of that is preliminary to a class discussion of why these rules hold and some elementary proofs. But the computer work helps them understand these rules and also seems to get them reasonably good at doing the manipulations by hand.

So I think that, like visual effects, using computers to do mathematics is a good educational strategy. But it is not the best way to use technology.

4.1.3. Using a Programming Language to Construct Mathematical Concepts

Let me begin with the biggest objection to having students write programs in order to learn mathematics. A couple of decades ago, this was a very popular idea and there was even a project (I believe its acronym was CRICISAM) to foster it. The trouble is that, in those days, programming was done in FORTRAN and there was so much extra effort in learning the programming language, dealing with bugs and other *non-mathematical* issues that any benefit that might accrue was canceled out.

I think this was a very accurate assessment. But that was then and this is now. The syntax of programming languages today can be quite simple and I hope that my illustrations in the section above on constructivism will suggest to the reader that rather sophisticated programs can be written with little syntactical difficulty. In fact, my experience with students is that although they continue to complain that the syntax gives them difficulty, I find that in just about every case the problem is either a mathematical concept that is not understood (like a function returning a function) or has to do with very inefficient work habits and methods of organizing material. My experience over the last decade or so has been that there are reasonable ways of using an appropriate computer language so that, in writing programs, syntax and system issues are minimized and the focus of the students can be almost entirely on the mathematics.

There are several reports of how we go about this and the reader can consult my WEB page at http://www.cs.gsu.edu/ẽdd/ for details. One thing I can say here is that our approach very much makes mathematics a laboratory course and, as Krantz suggests, we have the students meet, certain days of the week, in a computer lab where they work in cooperative groups on computer activities designed to foster the mental constructions we think they need to make in order to develop an understanding of the mathematics being studied. On the other days of the week, they meet in a classroom where the computer activities are discussed and work is focused on using the mental constructions they made in the computer lab to develop mathematical understandings.

4.2. Which is Best?

I think that all three of the above ways of using computers are effective in various ways, but I think that overall, the third is more effective than the other two. The most effective, however, is when all three approaches are used so that the student constructs mathematical concepts on the computer and then uses these constructs, or fancy versions of them found in a computer algebra system to do mathematics and to produce visualizations on a computer screen.

My argument for the effectiveness of writing programs in learning mathematics is two-fold. First of all, it relates to the theory in that specific mental constructs seem to arise out figuring out how to perform certain computer tasks. For example, if the student understands a certain mathematical procedure as an action or externally driven activity, then asking her or him to implement the procedure as a computer program tends to lead her or him to interiorize this action to a process. Moreover, I know of no more effective way of learning to encapsulate a process to an object than implementing the process as a computer program and then writing a program that uses that process as input and/or output. Thus writing a program that accepts two functions, constructs a program implementing the composition of those two functions and returns this program and then applying this tool in various situations helps a great deal in developing an understanding of functions as objects.

Again, there are reports of our research in which these effects are described and they can be found by consulting the above WEB page.

5. Applications and Pedagogy

Here we come to an issue on which I differ with my fellow reformers—and many traditionalists as well—perhaps more than on any other issue. Like Krantz, I feel that the wrong kinds of applications used in the wrong way can provide distractions to the mathematical issues on which we want the students to focus. Krantz refers to a problem about the destruction of trees in a tropical rain forest and suggests that a multitude of details about the situation tends to obscure the fact that what is needed here is to construct a function, take its derivative, set it equal to zero and solve. I think he is right and details that distract should be avoided, but I would go even further.

One argument is that is that a "real-world" context will make the problem more interesting for students and motivate them to use more energy in dealing with the situation. Unfortunately, one-person's real world is another's vague abstraction. This was brought home to me very sharply about 25 years ago when I was teaching (more or less traditionally) a unit on permutations and combinations. I asked the students how many starting line-ups could be made from a basketball squad with 12 members. I thought I was using a "real-world" example, but after class a student approached me and asked how many players there were on a starting line-up in basketball! It is true that this was at a "hockey school" and the student was a woman at a time when women were more or less excluded from basketball. Nevertheless, I realized that at least for this student, the application was not very helpful.

But even for students who do find a particular application context interesting, I question whether the resulting motivation really relates to the mathematics as opposed to the context—which can take them away from the mathematics! At the very least, I do not find any examples at all in the literature of research providing even a suggestion that using contexts that are interesting for students helps them understand the mathematics that *we* see in the context.

Given the lack of information that applications are helpful and the concern that the student will miss the mathematics in a context, or even be turned away from it, I think we should seriously reconsider our enthusiasm for the use of applications in helping students understand mathematics.

6. Some Specific Pedagogical Issues

Let me close with some brief comments about several issues that are, in fact, more at the heart of Krantz's book than perhaps are the matters I have been discussing at (probably too much) length. In spite of his very laudatory decision to relate his book to current issues of teaching methodology, research in learning and curriculum reform, Krantz has written a book on how to teach (traditionally). As such, he provides a lot of useful suggestions for beginners, and he presents his views on just about every educational issue one can imagine. In addition, he has provided a few of us with a platform on which to state our own views on some of these topics. Who can resist such an invitation? I will try to be brief.

Beginning with the most important, let me say that I agree with Krantz completely when he says that there is nothing essentially wrong with the content of any standard lower-division math course. I have a few quibbles such as too much emphasis on the analytic as opposed to the algebraic and geometric or the importance of adding courses that make mathematics a service course not only for the physical sciences, but the social and computer sciences. But for the most part, I think we have the content about right. This is my belief as a mathematician. I have also surveyed faculty who teach courses for which mathematics is a prerequisite and they tend to confirm this view. Perhaps the most telling argument is to take a look at the reform textbooks that are emerging and notice that the content is not really very different. After all, I am told that even the Babylonians asked their students about the rate of descent of the top of a ladder leaning against a building!

I think Krantz is right that students are not really ready for formal proofs until they have completed calculus and are taking one of the transition courses that have emerged in recent years. I think that forcing formalisms on students too early can contribute to our society's turning away from studying mathematics.

In my opinion, Krantz is too evenhanded about so-called objective examinations such as multiple-choice. No exam is really objective. Even a multiple choice examination makes a selection of material and what could be more subjective than making up the incorrect choices on a multiple choice exam? I think the overwhelming weight of arguments for such exams is the convenience of the person who grades the exam. A similar point can be made for timed exams. Would we really use them if it were not inconvenient to give students as much time (within reason) as they need? Do we really want to know only what they know so well that they have it on their finger tips and can produce it in high-pressure, emotionally tense situations? Don't we also want to know what they say when they have a chance to relax and reflect on a mathematical issue?

I think Krantz is very wrong when he says that "hard copy textbooks, more or less of traditional form, work. My 42 years of teaching experience tells me that traditional textbooks are essentially unread by students who use them mainly to find template solutions for problems that will be assigned for homework and given on the test. Moreover, I find that the greatest unanimity in all of education is found in the community's reaction to any attempt to vary from this norm. Such attempts are resisted, if not rejected, by students, publishers, and the overwhelming majority of faculty. This has been my experience with the textbooks I have written that are designed to support the ideas I have expressed here and I hear the same story from other writers of "different" textbooks. It seems to me that if we are to have real improvement in teaching and learning mathematics, either this situation has to change, or alternatives to textbooks must be found.

There is an orthodoxy about class size neatly expressed by Krantz who says that "We all know, deep in our guts, that small classes are a much more effective venue for learning than large classes." I sometimes wonder if what we really know deep in our guts is how to count and the salutary effect small classes will have on the job market. I am not sure we can't do really wonderful things with large classes. Personally, I feel that the most effective teaching I have ever done was with a calculus class in which there were 74 students. Not very large, but not exactly small either. My personal opinion is that using the methods referred to in this essay, I think we could learn to teach mathematics as well to classes of 200 as 20. Indeed, certain groups in France report success with new pedagogical strategies they are using in classes this large [1]. This is a largely unexamined issue. The research is, at best, mixed on the effects of class size on learning. In spite of the economics of the situation, I think we should take a good look at it and see if we can find pedagogical strategies that are both learning-effective and cost-effective.

Well, what better topic to end these ramblings with than the authority of the teacher and what is its source. Krantz suggests it comes from how you dress, from maintaining a certain distance, and from not being too chummy with the students. I think he has this completely wrong. Authority in teaching as in anything else comes from a strong, secure knowledge of and satisfaction with who you are and how you want to be, to dress, to talk and to move. This varies with people and the ones who get the most respect from their students are the ones who remain completely true to themselves, their nature and their personality. I have run many academic programs and often I am asked about a dress code. My answer always is: You dress to please yourself and anyone else you feel like pleasing.

Thank you, Steve, for this opportunity to spout off. Let's get a cup of coffee.

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Are We Encouraging Our Students to Think Mathematically?

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Introduction

In her 1990 paper, *Pedagogy and the Disciplines*, Ursula Wagener, of the University of Pennsylvania, describes a mathematics class:

> *A graduate student teacher in a freshman calculus class stands at the lectern and talks with enthusiasm about how to solve a problem: "Step one is to translate the problem into mathematical terms; step two is ..." Then she gives examples. Across the room, undergraduates memorize a set of steps. Plugging and chugging—teaching students how to put numbers into an equation and solve it—elbows out theory and understanding.*

The teacher in this classroom knows the subject matter. Her delivery and pacing are impeccable. Yet she teaches mathematics as a bag of tricks, rather than as an understanding of fundamental principles.

Is this an accurate description of the mathematics classes that we are teaching?

In order to help students learn to think mathematically, we first need to understand how they are thinking. Do students demand "plug and chug" from us? It is not so much that they demand it as they expect it. If we want to change their expectations, we must first understand their view of mathematics. Many students come to college with attitudes and expectations that may startle us.

Students' Expectations of First-Year Math Courses

One window into students' thinking processes is their comments on end-of-semester questionnaires. There are frequent comments about the instructor's clarity (or lack of it). Accessibility, sense of humor and accent are also important to students. They are also deeply concerned, and sometimes incensed, by issues of equity. Within a single course with common exams, some instructors are better than others, some give more handouts than others, some give review problems closer to the exam questions than others.

Another common complaint is that the instructor or the text didn't give the students enough help with the homework problems. For example:[1] "We spent hours doing some problems which [the instructor] didn't tell us how to do." Several would agree with the student who said the thing he liked least was that "...the examples provided did not help with homework." Another

[1] Source: Course evaluations.

suggested, "I wish we could have gone over one example like the homework problems ..." *before* we did the homework." Since many students didn't read math texts much in high school, but simply used them as a source of problems and examples, the kind of help they are asking for is a worked example, given in class or the text, which *closely* parallels the homework problem. To confirm this, the students in two courses at Harvard were surveyed and asked whether or not they agreed with the statement:

> *If you can't do a homework problem you should be able to find a worked example in the text that will show you how.*

Students in calculus gave it 4.1 out of 5 (where 5 indicates strong agreement); those in precalculus gave it a 4.7 out of 5. Further evidence of students' expectations came from the student who suggested that review problems should have the relevant section of the text listed after them in parentheses. When surveyed, his classmates agreed (4.2 out of 5 for calculus, 4.8 out of 5 for precalculus). More explicit still was the student who remarked that the best thing about the teaching in his section was that things were explained "in a cookbook fashion."

As instructors, we can learn a lot about how students think by listening to what they find difficult. The following two examples reminded me that although I feel it is important to know the meaning of one's computations, I do not always succeed in getting my point across.

- **Example: Wanting a 'Step-by-Step Approach.'** A student who had done badly on a Calculus II hour exam came to me complaining that he needed more step-by-step instructions on how to do the problems. Trying to narrow down the request, I asked what topics he already knew. One of them he was sure he knew, he said, was Euler's method. He had recently earned full credit on the following exam questions:

  > Consider the differential equation $dy/dx = x^2 + y$. **(a)** Use Euler's method with two steps to approximate the value of y when $x = 2$ on the solution curve that passes through (1,3). Explain clearly what you are doing on a sketch. Your sketch should show the coordinates of all the points you have found. **(b)** Are your approximate values of y an under- or over- estimate? Explain how you know.

  To check, I asked the student if he could draw a picture of the calculation he did for Euler's method. After a bit of thought, he said yes, he was sure he could draw such a picture. To check still further, I asked him what Euler's method was calculating. There was a *long* silence. Finally he said, rather hesitantly, that he thought it was the arc-length—the arc length along the polygonal curve.

Thus I would suggest that a more step-by-step method is not what was needed—but rather much greater attention on the part of both the instructor and the student to the meaning of the computations being performed. We should realize that exam problems which start, "Use such-and-such a method to do such-and-such," may not be testing all that we want to test. What if this exam question had been worded, "If $dy/dx = x^2 + y$ and $y(1) = 3$, estimate $y(2)$"? However, rewording this problem in this way would have caused some complaints, as students clearly agree (4.1 out of 5 in calculus and 4.6 out of 5 in precalculus) with the statement that:

> *A well-written problem makes clear what method to use to solve it.*

- **Example: 'Vaguely Worded' Problems.** Another student doing badly in Calculus II came to me after an exam to find out what to do. The problem was, as he described it, that although he understood the basic ideas, he couldn't apply them because of the "vague" way in which the problems were worded. The example he chose to illustrate this was the following question:

  > Alice starts at the origin and walks along the graph of $y = x^2/2$ in the positive x-direction at a speed of 10 units/second. **(a)** Write down the integral which shows how far Alice has traveled when she reaches the point where $x = a$. **(b)** You want to find the x-coordinate of the point Alice reaches after traveling for 2 seconds. Find upper and lower estimates, differing by less than 0.2, for this coordinate. Explain your reasoning carefully.

  The student had been unable to do this question because he hadn't realized that it was about arc length. He felt quite strongly that the wording of the question should have mentioned arc length specifically.

Again, the problem here seems to be how to teach students to do problems which do not explicitly ask for a certain computation, as well as how to get them to believe that such problems are reasonable.

How Do Students' Expectations Develop?

Most students come to college expecting new experiences. At a residential institution, we need to constantly remind ourselves that many students have never lived away from home before; many have never met students from different backgrounds, religions, or ethnic groups. It is easy to forget that many students have no idea what anthropology is and have never had the chance to take psychology or economics or visual arts. Most freshmen are eager for new experiences, but they are also insecure. Are they the admissions office's one mistake? Is it really worth their family making the financial sacrifices that are required to send them to college? One of the best ways for freshmen to reassure themselves that they are "doing OK" is to get decent grades—which often means *very* good grades, since they are often used to all A's. This makes them want to take some courses which are familiar and predictable so that they can be sure they will do well. This is particularly true in math and science where the grading and workload are inclined to be tougher than in other fields.

Undergraduates are busy people. It is easy for us to underestimate the pressure to join extracurricular activities and our students need to hold a job (or even two). Consequently, undergraduates prize courses and instructors which do not take too much of their time outside of class. A course in which they are expected to read about some topics on their own may have students struggling to get out of it, not because they can't do the work but because it takes too much time and they don't want to have to "teach the course to themselves." The feeling that everything "ought" to be covered in class is reflected in student reactions to mathematics. As an example, one student said that the best thing about the teaching in his section was the fact that

> *It was possible for me to learn the material without studying on my own too much if I paid attention in class.*

Thus most undergraduates, like most faculty, have more to do than they can reasonably fit into their schedules and tend to look for short-cuts. As an analogy for the way in which some students go about learning mathematics, imagine what it takes to become proficient at an office manager's job. Most undergraduates regard learning mathematics in much the same way a mathematician might regard learning such a job: as something you have to be shown how to do.

This view of mathematics is partly reasonable—we don't expect students to come up with the idea of a derivative by themselves, or to figure out the Fundamental Theorem of Calculus on their own. However, carried to extremes it can lead to absurd results. Consider, for example, the student (who had already taken BC Calculus) who was complaining rather loudly that he shouldn't be asked to suggest formulas for functions whose graphs he had been given. He said that he "did graphs" in the other order: If we gave him the formula, he'd draw the graph, but he wasn't doing this. When it was suggested that he might need to find formulas to fit lab data in his intended major (chemistry), he announced that he'd done experiments before and that one didn't *ever* need to do such a thing. It is important to realize that the vehemence that leads students to dig in their toes as completely as this one is born, at least partly, out of terror, not out of a real desire to be closed-minded.

A certain amount of predictability is entirely appropriate and necessary. Too much, however, means that it is easier for students to memorize how to do a long list of "types" of problems than to learn the basic ideas of mathematics. Many of our students are sufficiently diligent and sufficiently scared of not performing well that they will willingly master a very long set of problem-solving procedures, even by memorization. For example, while trying to figure out how to prepare for a calculus final, one student asked me if he should do the review problems over and over again until he could do them without looking at the solutions. In this way they differ greatly from the "creatively lazy" professional mathematician who would much rather figure out how it all works than memorize anything.

Most students coming to college know, at least in theory, that learning mathematics involves developing understanding. Many of them, often on their own, have developed an understanding of some topics. However, few of them have actually taken courses where an understanding was really *required*—in virtually every case, it was possible to do well just by learning to do all the types of problems shown in class. Consequently, asking students to do problems which have not been modeled for them in class or in the reading is inclined to strike them as unfair.

The Instructor's Role

There is ample evidence that the students' views of mathematics are often startlingly different than ours when they start college. What can be done about it? Is encouraging students to think mathematically part of our job, or should we just work with those students who are naturally inclined this way?

I suggest that we may be letting our students down if we do not try to broaden and deepen their thinking. It is, of course, much easier for us to teach a course in which we focus on template problems and algorithms. It's even possible to put the burden on the students by saying that they won't accept anything else. But is that really true? Perhaps, as George Rosenstein, Franklin and Marshall, suggests,[2] we have played a role in allowing these expectations to become established:

> *I'm convinced there has been a conspiracy between math teachers and math students. The terms are that the teachers can do whatever they want in class, but will ask for only the well-practiced or routine on the exams. In return, the students will be cooperative and diligent at learning the manipulative or template material that is stressed on homework assignments and quizzes.*

Our biggest challenge as teachers is to understand our students' thinking patterns well enough that we can affect them. Learning to think more independently is a difficult, often frightening, process for students. Thus, besides gaining an understanding of our students' thinking, we need to understand their feelings well enough to gain their trust.

How Should Our Students' Views of Mathematics Affect Our Teaching?

Before considering how to react to our students' view of mathematics, we need to consider what is meant by understanding. Here, too, there is a difference between professional mathematicians and first year undergraduates. In most students' minds, understanding is being able to visualize (or otherwise internally represent) concepts and the relationships between them. The notion which mathematicians might call, or include in, understanding—knowing the theoretical and logical connections between concepts—is not what most students mean by understanding. For most students, the visual form of understanding precedes the more theoretical version. Thus, to take students toward a rigorous thinking, we must first establish a solid intuitive, visual understanding.

[2]From Project CALC Newsletter, October 1991.

Each department and each faculty member needs to decide exactly how to approach students' views of mathematics. Not to challenge these views at all is not doing our students justice, as well as not in the best interests of the profession. However, challenging them too much alienates students from the subject and teaches them little. The most useful guide to how much to challenge them is a robust understanding of students. To acquire this, each instructor is well advised to use every course they teach as an opportunity to learn about their students. Some techniques that may be useful:

1. Include problems on tests that ask students to sketch or explain. Grade them yourself, or look them over before handing them back.

2. Ask students to read some mathematics and summarize what they have read in a paragraph—and find time to read, or skim, the paragraphs.

3. Arrange an e-mail discussion group for a course. Look over the postings every few days.

4. A particularly good way to gain insight into what students are thinking is the "one-minute paper," advocated by Richard Light at Harvard. At the end of class ask students to take a piece of paper and write on it:

- The most important thing learned that day.
- The most confusing thing that day.
- One question they have that remains unanswered.

The pieces of paper are handed in as the students leave the room; the instructor reads through them before planning the next class. The answers to these questions can often be illuminating—and occasionally devastating—to the instructor. It becomes easier to see what the students are thinking, and addressing their misconceptions becomes more urgent. Acknowledging and acting on the responses can markedly improve communication with the class.

Conclusion

Successful teaching involves knowing what your students are thinking. However, it is often hard for instructors to "hear" students—especially as the students' thoughts are often not similar to their own. Consequently, any efforts to listen to students' thinking about mathematics are likely to improve one's teaching.

Big Business, Race, and Gender in Mathematics Reform

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The mathematics reform movement may have positive attributes, but that is not what this appendix is about. This essay is divided into three sections, each taking a critical view of what has come to be called "mathematics reform." Rather than attempting an abstract definition of this term, I cite the principal documents and leaders of the reform movement on particular issues. The fault line separating the mathematics reform movement from its critics is nowhere more volatile and portentous than in California. The third and final section of this appendix is devoted to a short history of the conflict over mathematics reform in that state, with a focus on the controversial California mathematics standards. This set of standards has received widespread praise from prominent mathematicians and strong opposition from the mathematics reform community. As explained in the last section, this conflict helps to define, in *practical* terms, the mathematics reform movement.

The second section challenges assumptions about ethnicity and gender in the reform movement. Multiculturalism and mathematics for "all students" are recurring themes among reformers. Prominent reformers claim that learning styles are correlated with ethnicity and gender. But reform curricula, while purporting to reach out to students with different "learning styles", actually limit opportunities. Fundamental topics, including algebra and arithmetic are abridged or missing in reform curricula without apology.

Big Business and the mathematics reform movement have at least one thing in common. They both militate for more technology in the classroom. Calculators and computers are regular features in reform math curricula, and technology corporations routinely sponsor conferences for mathematics educators. The confluence of interests and the resulting momentum in favor of more technology is the subject of the first section.

Technology, Reform, and the Corporate Influence

The 1989 report "Everybody Counts" warned:

> In spite of the intimate intellectual link between mathematics and computing, school mathematics has responded hardly at all to curricular changes implied by the computer revolution. Curricula, texts, tests, and teaching habits—but not the students—are all products of the pre-computer age. Little could be worse for mathematics education than an environment in which schools hold students back from learning what they find natural. [17]

The imperative to integrate technology into the classroom goes far beyond mathematics courses. President Clinton calls for "a bridge to the twenty first century . . . where computers are as much a part of the classroom as blackboards." [18] Presently, four-fifths of U.S. schools are wired to the Internet and the rest are not far behind. Remonstrations by well-placed technology experts and educators, based on educational considerations, seem to warrant no delays [6], [18]. A 1996 report by the California Education Technology Task Force, a group dominated by executives in high tech industries, called on California to spend $10.9 billion on technology for schools before the end of the century. The task force claimed that "more than any other single measure, computers and network technologies, properly implemented, will bolster California's continuing efforts to right what's wrong with our public schools." [10]

A corporate perspective is also sweeping American universities with computer technology paving the way. The number of virtual universities and virtual courses is increasing exponentially. In 1997 there were 762 "cyberschools", up from 93 in 1993 and more than half of the nation's four year colleges and universities have courses available "off site" [4]. The second largest private university in the U.S., the University of Phoenix, offers on-line courses to 40,000 students from a faculty with no tenure. Other examples[1] and a recent history of technology and the corporatization of universities may be found in David Noble's interesting essays [16].

Will the computerization of schools improve education? The Los Angeles Times reports that "many critics worry that education policy is increasingly being driven by what companies have to sell rather than what schools need . . . Computer companies want more technologically savvy consumers, for example, to increase the penetration of computers beyond the 40% of homes in which they are now found. And they argue that increased use of technology in schools will help fill a growing shortage of computer literate workers." [10]

Corporate foundations regularly fund mathematics reform projects, as for example, the "Exxon Symposium on Algebraic Thinking" for the Association of Mathematics Teacher Educators Conference, held in January 1998, with Texas Instruments hosting one of the dinners. Conversely mathematics reformers embrace a corporate vision of education, which includes the de-emphasis of basic skills and a greater reliance on technology. Consider, for example, the following promotional material for the K-6 curriculum MathLand from Creative Publications:

> Business leaders have expressed interest in changes in education as

[1] During the mid-1990's the California State University administration initiated an unprecedented partnership with technology giants Microsoft, GTE, Fujitsu and Hughes Electronics. The joint venture, called the California Educational Technology Initiative, or CETI, will, if implemented, wire up the 23 campuses of the CSU with state-of-the-art telephone and computer networks, as well as invest billions of dollars in education-related electronics. By the Spring of 1998, a dozen CSU campus faculty senates passed resolutions asking for delays and criticizing the merger. The California State Student Association passed its own resolution denouncing CETI and opposing any "privatization of the California State University as a whole." Microsoft and Hughes subsequently pulled out, but CSU Chancellor Reed continues to seek new corporate partners. The implications of such a partnership are not fully worked out, but incentives for the faculty to market computer products to students, and the creation and marketing of courseware have been seriously considered.

> well—changes that go beyond what a traditional standardized test can measure. Recently, the US Departments of Labor and Education formed the Secretary's Commission on Achieving Necessary Skills (SCANS) to study the kinds of competencies and skills that workers must have to succeed in today's workplace. According to the SCANS report What Work Requires of Schools: A SCANS Report for America 2000, business leaders see computation as an important skill, but it is only one of 13 skills desired by Fortune 500 companies. These skills are (in order of importance): teamwork, problem solving, interpersonal skills, oral communication, listening, personal development, creative thinking, leadership, motivation, writing, organization skills, computation, and reading. [15]

The California Mathematics Council (CMC), an affiliate of the National Council of Teachers of Mathematics, boasts 12,000 members. In an open letter to the California Board of Education dated April 17, 1996, the CMC included the same ordered list of basic skills with reading and computation given last. Citing unspecified "educational research" and "neuro-biological brain research", the CMC letter endorsed the direction of the 1992 reform-oriented California Mathematics Framework and added:

> Equally impressive is that these changes in the way we teach mathematics are supported by the business community. What Work Requires of Schools: A SCANS Report for America 2000 concludes that students must develop a new set of competencies and new foundation skills. It stresses that skills must be learned in context, that there is no need to learn basic skills before problem solving, and that we must reorient learning away from mere mastery of information toward encouraging students to solve problems.

Learning in order to know must never be separated from learning to do. Knowledge and its uses belong together (A SCANS Report) [2]

The NCTM Curriculum and Evaluation Standards also recommends that "appropriate calculators should be available to all students at all times" and "every student should have access to a computer for individual and group work." Reform texts place little restriction on technology. The second edition of the Harvard Calculus text instructs that students "are expected to use their own judgment to determine where technology is useful." [7] The 1992 California Mathematics Framework recommends that calculators be available at all times to all students, including Kindergarten students, and asks, "How many adults, whether store clerks or bookkeepers, still do long division (or even long multiplication) with paper and pencil?"

None of the above is intended to suggest that technology should not be used in mathematics classes. Nor do I suggest any kind of conspiracy theory. I have incorporated the (limited) use of computers in some of my own classes at California State University, Northridge, and I agree with almost all of Professor

Krantz' balanced discussion in Section 1.10. My only reservation is Professor Krantz' suggestion that an entire "lower-division mathematics curriculum [should] depend on `Maple` (or `Mathematica`, or another substitute) and that [students] need to master it right away." This seems to me to be premature. It might be appropriate at some point, but a compelling curriculum with this feature should be presented, vigorously reviewed, and thoroughly tested on real students first.

The use of technology in mathematics education should be considered against the backdrop of extremely powerful business interests which seek to create new consumers of technology. Incorporating ever more technology into the classroom may or may not be consistent with good educational practices. Large-scale implementations of technology in the classroom receive tremendous momentum from funding agencies—at times, far beyond what the results merit. With the huge sums of money involved in computerizing education, the educational merits of technology are rarely discussed. For politicians and entrepreneurs, no justification is necessary, but educators should demand clear evidence of the beneficial effects of technology before it is incorporated in classrooms.

The calculator is one of the staples of the reform mathematics movement from Kindergarten through calculus and beyond. Mathematics instructors, including calculus teachers, regularly allow students to use calculators on examinations and contort their tests to avoid giving points for mere button pressing skills. I agree with Professor Krantz' assertion in Section 1.10 that "if a student spends an hour with a pencil—graphing functions just as you and I learned—then there are certain tangible and verifiable skills that will be gained in the process." I don't think it is unreasonable to require students to demonstrate these and other skills on examinations without calculator assistance.

At the elementary school level, arithmetic is a victim of technology in the reform curricula. Long division in particular is frequently a target for elimination. For example, long division with more than single digit divisors was consciously eliminated from the proposed California math standards by the Academic Standards Commission, and the California Mathematics Framework makes it clear that "clerks or bookkeepers ... do [not do] long division ... with paper and pencil." In addition to sharpening estimation skills, mastery of the division algorithm is important for understanding the decimal characterization of rational numbers, a middle school topic, as well as quotients of polynomials and power series in later courses.

In a society that worships technology, it is all too easy to surrender the integrity of sound traditional curricula to machines, their corporate vendors, and reform-evangelists.

Gender, Race, and Ethnicity in the Reform Movement

One of the themes of the mathematics reform movement is that women and members of ethnic minority groups learn mathematics differently than white males. The thesis that learning styles are correlated with ethnicity and gender is widely accepted in education circles and its validity is not assumed to be restricted to mathematics. One example of this ideology occurred when the Oakland School Board resolved that Ebonics is genetically based [19]. Mainstream views from the academy are similar. In a well-referenced study on how African Americans learn mathematics, published in the Journal for Research in Mathematics Education, one finds [14]:

> Studies of learning preferences suggest that the African American students' approaches to learning may be characterized by factors of social and affective emphasis, harmony with their communities, holistic perspectives, field dependence, expressive creativity, and nonverbal communication . . . Research indicates that African American students are flexible and open-minded rather than structured in their perceptions of ideas . . . The underlying assumption is that the influence of African heritage and culture results in preferences for student interaction with the environment and that this influence affects cognition and attitude . . .

The Journal of American Indian Education devoted an entire special issue to the subject of brain hemispheric dominance and other topics involving Native American learning styles. Included is a reprint of Dr. A. C. Ross's paper, "Brain hemispheric functions and the Native American," that asserts Native Americans are "right-brained". Ross explains that the "functions of the left brain are characterized by sequence and order while the functions of the right brain are holistic and diffused." Elaborating, he maintains that "left brain thinking is the essence of academic success as it is presently measured. Right brain thinking is the essence of creativity." Citing earlier research, Ross concludes that "traditional Indian education was done by precept and example (learning by discovery) . . . creativity occurs in the learning process when a person is allowed to learn by discovery. Evidently, traditional Indian education is a right hemispheric process." [9] The final article in the same journal takes issue with this point of view and laments that "a veritable right-brain industry has developed" and warns of the dangers to Indian education by characterizing this entire ethnic group as right brained.

The view that women and minority group members learn differently from white males is far from marginal within the mathematics reform movement. A radio interview of NCTM President Jack Price, independent textbook publisher John Saxon, and Co-Founder of Mathematically Correct, Mike McKeown, occurred on April 24, 1996. The KSDO radio show on Mathematics Education, hosted by Roger Hedgecock, was held in conjunction with the annual meeting of the National Council of Teachers of Mathematics, in San Diego that year. During the interview, President Price asserted:

> What we have now is nostalgia math. It is the mathematics that we have always had, that is good for the most part for the relatively high socio-economic anglo male, and that we have a great deal of

> research that has been done showing that women, for example, and minority groups do not learn the same way. They have the capability, certainly, of learning, but they don't, the teaching strategies that you use with them are different from those that we have been able to use in the past when young people, we weren't expected to graduate a lot of people, and most of those who did graduate and go on to college were the anglo males. [13]

The reform movement presupposes that broad classes of non-white males learn "holistically", that mathematics should be integrated with examples and connected as widely as possible to other human endeavors. Algebra and arithmetic are particularly short changed as "mindless symbol manipulation" and "drill and kill." To cite one typical example of this, a mathematics educator wrote on the Association of Mathematics Teacher Educators listserve, "I know this may come as a shock to some mathematics professors out there, but few students find manipulating x and y engaging."

It is clear that proponents of reform are acting out of a sincere desire to improve mathematics education for all students. But the mathematics community should be suspicious of trends which draw legitimacy from racial or gender theories of learning.

No one disputes that culture plays a role in academic achievement. The Los Angeles Times published a special report entitled, "Language, Culture: How Students Cope" as part of a three day series of special reports on education [11]. The Times report explicitly discounts any link between race and ability, but acknowledges that "ethnic differences [in academic accomplishments] remain, even after accounting for income, parent education or language a student speaks at home." High achievement by Asian American students is a result of hard work and a strong emphasis on the importance of education, and this contrasts sharply with the "complacency that hampers so many of California's white students, who have shown a sharper drop on reading scores this decade than either blacks or Latinos." "The burden of acting white" is a theory that African American students "resist schooling to protect their self-image and distinguish themselves from a majority culture that too often devalues their abilities." Many Latinos, for cultural and economic reasons, may see pursuing an education as selfish, since getting a job instead would contribute directly and immediately to family members [11].

Investigating cultural reasons for differences in academic achievement is quite different from proposing that members of different ethnicities and genders actually learn mathematics in different ways. The latter point of view is especially serious when it leads to new, watered-down mathematics curricula.

There should be no doubt that minority students can thrive in traditional programs. Take the case of Bennett-Kew Elementary School in Inglewood, California. According to Principal Nancy Ichinaga, 51% of the students are African-American and 48% are Hispanic (mostly immigrant with Limited English Proficiency). Approximately 70% qualify for subsidized lunches. Below are its 1997 California Achievement Test results, with Normal Curve Equivalent scores (similar to percentiles):

Grade	1	2	3	4	5
Math	62	79	81	75	68

Bennett-Kew believes in high, explicit standards for all students. The mathematics standards are not just year-by-year, but month-by-month. There is regular diagnostic testing of student progress and immediate remediation. The school is committed to direct instruction and does not use newer books. While discussing the mathematics reform movement with me, Principal Ichinaga remarked, "Reform is for the birds."

The traditional approach to teaching calculus used by the legendary teacher Jaime Escalante is another example of minority students thriving in a traditional mathematics program. In 1974, Escalante took a job teaching basic mathematics at Garfield High School which was in danger of losing its accreditation because discipline and test scores were so bad. Five years later, insisting that disadvantaged and minority students could tackle the most difficult subjects, he started a small calculus class. The effect was to raise the curriculum for the entire school. In 1982, 18 of his students passed the Advanced Placement calculus exam. This was the subject of the movie, *Stand and Deliver*. Working with his fellow calculus teacher Ben Jimenez, and Garfield Principal Henry Gradillas, Escalante sent ever increasing numbers of students to leading universities with AP calculus credit.

By 1987, Garfield High School had more test takers than all but four high schools in the United States. The number of test takers reached its peak of 143 students in 1991, the year Escalante left Garfield. The passage rate was 61%. The numbers have declined ever since. By 1996 there were only 37 test takers with a passage rate of 19%. It is interesting that former Principal Gradillas' career declined after the spectacular successes of his high school. After finishing his doctorate in 1987, he "expected to be given an important administrative job that would help spread the school's philosophy to other parts of Los Angeles. Instead he was told to supervise asbestos inspections of school buildings. District officials denied they were punishing him, but one said privately that Gradillas was refused better assignments because he was considered 'too confrontational'." [21]. Rather than studying his effective methods, Escalante is shunned by the mathematics reform community. The disapproval is mutual. According to Escalante, "whoever wrote [the NCTM math standards] must be a physical education teacher." [3]

> The calculus reform movement is inextricably linked to the K–12 mathematics reform movement. Consider, for example, the following statement from the preface of the first edition to the Harvard Calculus text [7]:
>
> We have found this curriculum to be thought-provoking for well prepared students while still accessible to students with weak algebra

> backgrounds. Providing numerical and graphical approaches as well as the algebraic gives the students several ways of mastering the material. This approach encourages students to persist, thereby lowering failure rates.

Lower failure rates at the cost of eviscerating the algebra component of calculus is harmful to students of all ethnicities and both genders. Algebra and arithmetic are consistently de-emphasized in reform curricula in exchange for the more "holistic" calculator assisted "guess and check" routine. The entire reform program mortgages future opportunities to attend to the immediacy of high failure rates. The de-emphasis of algebra in reform calculus justifies and caters to the K–12 reform mathematics program.

Calculus proofs and even definitions require students to be competent in algebra. Calculus reform texts tend to relegate both of these to appendices, sparing students the necessity even to turn a few pages in order to avoid them. Instructors who wish to include definitions, such as the definition of a limit and/or a few proofs, must overcome additional psychological resistance because of the location of these topics in the textbooks. When a proof or definition is placed in an appendix, it sends the message to the student that the topic is not important and may be safely skipped.

The emergence of these trends at a time when greater numbers of previously under-represented students are attending universities should cause some reflection within the mathematics community. Are we expecting less of these students? If so, is it because they learn mathematics differently from students of an earlier era, or is it because their mathematical preparations are deficient? I think it is the latter, and I believe that the mathematics community would do well to purge itself of any hidden assumptions that non-Asian minority students learn mathematics differently from anybody else. The focus should be on raising the level of mathematics education in K–12, not on how best to lower it in the universities.

The Politics of Mathematics Reform in California

Nowhere has the conflict over mathematics education reform been more contentious than in California. California led the United States in institutionalizing K–12 mathematics reform. The 1992 California Mathematics Framework is based on the 1989 NCTM Standards and has served as a guide for politically powerful reformers, like the California Superintendent of Instruction, Delaine Eastin (elected in 1994), as well as countless specialists in the state's Colleges of Education who have used it as course material for K–12 student teachers. But California's commitment to the principles of mathematics reform predates the NCTM Standards. For example, one finds in the 1985 "California Model Curriculum Standards, Grades Nine Through Twelve":

> The mathematics program must present to students problems that utilize acquired skills and require the use of problem-solving strate-

> gies. Examples of strategies that students should employ are: estimate, look for a pattern, write an equation, guess and test, work backward, draw a picture or diagram, make a list or table, use models, act out the problem, and solve a related but simpler problem. The use of calculators and computers should also be encouraged as an essential part of the problem-solving process. Students should be encouraged to devise their own plans and explore alternate approaches to problems.

The educational philosophies behind the mathematics reform movement are canonical in America's colleges of education and have been for most of this century [8]. The broad principles of reform have been institutionalized in California state documents for well over a decade and have taken root in the schools. Reform curricula based on these principles are ubiquitous in California's elementary schools. The controversial curriculum, "MathLand", for example, has been adopted by 60% of the state's public elementary schools, according to its publishers [20], and there are many other similar curricula widely in use. Secondary mathematics curricula such as Interactive Mathematics Program and College Preparatory Mathematics originated in California and are widely used throughout the state at the time of this writing. The alignment of these and other self-described reform curricula with the NCTM Standards seems to be uncontested. Indeed, much of the development and implementation of these curricula has been funded by the National Science Foundation and other powerful, reform dominated institutions. In particular, MathLand, perhaps the worst of all reform curricula, has been promoted through the NSF funding.

California is experiencing a backlash at the grass-roots level against the general education reform movement (including Whole Language Learning and "Integrated Science"), and mathematics reform in particular. Reacting to the de-emphasis of arithmetic and algebra in the reform curricula, and the over-reliance on calculators, parents' education organizations have emerged all over the state, several with their own web sites containing material starkly critical of "reform math" or "fuzzy math". I am associated with the largest and best known of these groups, "Mathematically Correct".

Of particular concern to parents and teachers critical of the reform movement is the lack of accountability and measurable standards of achievement in the schools. "Authentic assessment" in place of examinations with consequences, and little if any importance placed on student discipline and responsibility in the reform literature, help to make reform math an object of ridicule among vocal parents' groups. It is noteworthy that all parties acknowledge the importance of better teacher training.

The conflict between the mathematics reform movement, on the one hand, and parents' organizations combined with a significant portion of the mathematics community, on the other hand, reached a turning point in December, 1997. At that time the California Board of Education rejected the reform-oriented draft standards from one of its advisory committees—the Academic Standards Commission—and, with the help of Stanford mathematics professors

Gunnar Carlsson, Ralph Cohen, Steve Kerckhoff, and Jim Milgram, developed and adopted the California Mathematics Academic Content Standards [1].

Unlike the Academic Standards Commission proposal, these new standards made no pronouncements about teaching methods, only grade- level benchmarks. The reaction from the California mathematics reform community against this lack of coercion was swift and harsh. Their response was to claim that the official math standards, written by the Stanford mathematicians, lowered the bar. Turning reality on its head, State Superintendent Delaine Eastin charged, "[The State Board of Education Standards] is 'dumbed-down' and is unlikely to elicit higher order thinking ..." Judy Codding, a member of the Academic Standards Commission and the powerful National Center on Education and the Economy put it bluntly when she said, "I will fight to see that [the] California Math Standards are not implemented in the classrooms" [22].

Other Reformers with national stature echoed the outrage. Luther Williams, the National Science Foundation's Assistant Director for Education and Human Resources, wrote a retaliatory letter to the California Board of Education widely interpreted as threatening to cut off funding of NSF projects in California. The lead story in the February 1998 News Bulletin of the NCTM, *New California Standards Disappoint Many*, began with the sentence, "Mathematics education in California suffered a serious blow in December." The article quotes a letter from NCTM President Gail Burrill to the president of the California Board of Education that included the statements: "Today's children cannot be prepared for tomorrow's increasingly technological world with yesterday's content ... The vision of important school mathematics should not be one that bears no relation to reality, ignores technology, focuses on a limited set of procedures, ... California's children deserve more." Presumably the accusation that technology is ignored refers in part to a policy decision of the California Board of Education that state-wide exams based on the new math standards will not include the use of calculators—a serious blow to the corporate/reform ideology.

Joining the reform math community, the state-wide chairs of the Academic Senates of the UC, CSU, and California Community College systems, none of whom were mathematicians, issued a joint statement condemning the adoption of California's math standards and even suggested that "the consensus position of the mathematical community" was in opposition to the new standards, and generally in support of the rejected, reform- inspired draft standards written by the Academic Standards Commission.

In opposition to the reform community and in support of California's new math standards, more than 100 California college and university mathematicians endorsed an open letter addressed to the Chancellor of the 23-campus California State University system. The open letter disputed the existence of such a consensus and urged the Chancellor to "recognize the important and positive role California's recently adopted mathematics standards can play in the education of future teachers of mathematics in the state of California." Among the endorsing mathematicians were several department chairs and many leading mathematicians [12].

Further contradicting the reformers' claims against California's math standards, Ralph Raimi and Lawrence Braden, on behalf of the Fordham Foundation, conducted an independent review of the mathematics standards from 46 states and the District of Columbia, as well as Japan. California's new board-approved mathematics standards received the highest score, outranking even those of Japan [5].

The sharp conflict over the California math standards defined, in practical terms, the mathematics reform movement. Reformers denounced the state's standards in public forums and the press, while traditionalists and critics of reform defended the standards. Based on purely mathematical considerations, the board-approved California standards are easily seen to be superior to the rejected, reform-oriented version offered by the Academic Standards Commission. A careful and well-written comparison these two sets of standards by Hung-Hsi Wu is available on the Mathematically Correct web site [22].

The extent to which the California math standards will be taken seriously by school districts is difficult to predict. The superintendent of the Los Angeles Unified School District, the second largest school district in the U.S., admonished LAUSD personnel to take no action to implement the new standards, arguing instead that the already existing LAUSD standards were superior. A refutation and insightful comparison of the LAUSD math standards with the California standards was developed by Jim Milgram, Mathematically Correct Co-Founder Paul Clopton, and others. It is also available on the "Mathematically Correct" web site [13]. The LAUSD K–12 math standards are vague and repetitive, trigonometry is completely missing, and third graders are encouraged to use calculators.

Opposition to California's mathematics standards from reform leaders continues as of this writing. Former NCTM president Jack Price wrote in a letter published by the Los Angeles Times on May 10, 1998:

> ... if the state board had adopted world-class mathematics standards for the 21st century instead of the 19th century, there would have been a great deal of support from the 'education' community.

This sententious observation encapsulates the topics discussed in this essay. For the reformers, "world-class mathematics standards for the 21st century" eluded the Stanford mathematicians who wrote California's 1998 math standards. Missing are the greater emphasis on technology—an end in itself—and pedagogical directives harmonious with the reified "cognitive styles" of the racially diverse populations of the 21st century. The "19th century" arithmetic, algebra, geometry, and trigonometry highlighted in California's 1998 standards will have diminished value in the postmodern epoch of technological wonderments envisioned by math reformers.

Perhaps the academic community should consider whether the discipline of mathematics education—much more so than mathematics—needs fundamental alterations for the 21st century.

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Will This Be on the Exam?

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1. The Proof of the Pudding . . .

Many of us in academe remember with pleasure our experience, as students, of the final exam period: the few days before each exam, studying the textbook, memorizing facts and formulas, grasping anew the basic concepts, and constructing for the first time a complete picture of the course. We went into each exam with a sense of anticipation, and left with the pleasure of a job well done. As a teacher, I enjoy constructing final exams, balancing easy questions with hard, rote skills with conceptual understanding, theory with applications; providing some lucky student with the experiences I had.

The tests and exams which determine a student's final grade form the definitive statement, from the student's point of view, of what is in a course.[1] Although it is important for instructors to think about standards, skills, conceptual understanding, and applications, it is the exams that state the instructor's true expectations, and it is the student's performance on those exams that indicates how well he or she has met those expectations. There is often a serious contradiction between the envisioned syllabus, the subject of discussions in textbook committees, and the real syllabus, represented by the final exam. The question that forms the title of this article, so often asked by our students, would be a good question for us to ask ourselves whenever we are discussing the curriculum. Our discussions would be more cogent if we started at the end of the course, so to speak, with a frank admission of what we really expect our students to be able to do, rather than at the beginning, with the syllabus or the textbook's table of contents. It doesn't much matter what you teach or how you teach it if the students can study for your final by memorizing soon-to-be-forgotten procedures.

2. A Test for Tests

A couple of years ago I obtained some sample final exams from the mathematics department of a prestigious university. One was a calculus exam, which, out of curiosity, I gave to `Mathematica`. Overall, it did very well, although on some questions it would not have earned full marks, as when it correctly stated that an infinite series diverged, but failed to give the test that confirmed this. Perhaps it did not show as much work as a grader would have wished, but it would be easy to customize it to do so (indeed, there are calculators that will 'show the steps' in symbolic differentiation and integration). Most strikingly, there was not a single question that did not have a corresponding `Mathematica`

[1] Although I will frame this discussion in terms of exams, my comments apply to any other form of final assessment.

command. Barely one asked for a demonstration of conceptual understanding or an ability to select the appropriate tool (beyond knowing whether to type `Integrate[]`, `Differentiate[]`, or any others of a small range of commands). Furthermore, barely any explanation of the answers was requested (although perhaps the need for explanation was tacitly understood).

What conclusions can one draw from this? I would like to avoid drawing conclusions on the issue that has been most bitterly contested, the issue of how technology should be used in teaching and examining students, and what technical skills remain vital in the age of technology. Rather, I would like to point to a conclusion that is indisputable, no matter what one's attitude to technology. The fact that a computer algebra system was able to perform so well on this exam is an indication of its low intellectual content. Imagine an English final exam, at the college level, that consisted of nothing but spelling and the elementary grammar built into current grammar-checkers.

It might be argued that there is nothing wrong with such exams in mathematics, or with the courses that go with them. There is indeed a tradition of techniques courses, which aim explicitly to teach nothing more than mathematical techniques for engineers and scientists. I have heard it argued that freshman calculus should properly be regarded as such a course. Whether or not such courses still make sense in the age of technology is a matter of often vociferous debate, and I don't want to get into that debate right now. Rather, let me, for the rest of this discussion, limit myself to courses which are intended to be more than techniques courses.

For such courses, I'd like to propose a test for evaluating exams, inspired by my `Mathematica` experiment. When you write an exam, ask yourself:

Can all of the questions be answered satisfactorily using purely mechanical procedures?

By mechanical procedures, I mean not only computer algebra systems, but problem templates, pencil and paper algorithms, anything that a computer could be programmed to do. Common sense applies: I exclude Rube Goldberg contraptions narrowly designed for one specific question. If all of the exam, or a large part of it, can be handled mechanically, with output that would be acceptable to you as a grader, then it doesn't pass my test.

I am not arguing that no exams should be limited to mechanical skills, but that there should be some exam that isn't so limited, some point where the course, as represented in the questions you grade your students on, should rise to a higher intellectual plane. Furthermore, it is quite possible to reach this higher plane and still allow computer algebra systems on the exam. Such exams, however, would at some point request mathematical reasoning and verbal explanations, which cannot be supplied by the computer.

Finally, I want to draw attention to the stipulation in my criterion that questions be answered satisfactorily. In applying the criterion, it is important to consider not only the text of the questions on your exam, but your grading standards. The first example in the next section illustrates this point.

3. Good Questions and Good Answers

What are the consequences of my proposed criterion? Here are some general principles of question design and grading standards that can drawn from it. Again, let me emphasize that I do not mean to exclude mechanical questions altogether; however, in what follows, I'll give examples of questions that, if included on an exam, would raise it above the purely mechanical level. I have used all these questions on final or midterm exams in courses that I have taught, with the exception of the first one, which I use in a handout to students on how to write mathematics, and the last one, which I made up for this article. To better illustrate my principles, I have chosen easy questions. At least, they should be easy; however, students do not find them so. If you find this hard to believe, I encourage you to try them yourselves, under the following conditions: **a)** Give them as written, without any added hints or suggestions. **b)** Do not prepare or warn the students in any way.

Require answers to be complete sentences. One thing that computers can't do very well (yet) is think about why they did what they did, and write about it in convincing English prose. On the other hand, this is something we want our students to be able to do. Consider the following extremely easy question.

- *Question.* Find the equation of the line through the points $(-1, 0)$ and $(2, 6)$.

Here are two answers, using the same method, but written very differently. The method might not be one you prefer to teach; however, it is not the method but the difference in writing which illustrates my point.

- *Answer 1.*

$$m = \frac{6-0}{2-(-1)} = 2$$
$$y = 2x + b$$
$$0 = 2(-1) + b$$
$$b = 2$$
$$y = 2x + 2$$

- *Answer 2.* The slope of the line is

$$m = \frac{6-0}{2-(-1)} = 2,$$

so the equation of the line is $y = 2x + b$, for some constant b. Since $(-1, 0)$ is on the line, $0 = 2(-1) + b$, so $b = 2$, and therefore the equation is $y = 2x + 2$.

Answer 1 demonstrates the ability to follow an algorithm; Answer 2 demonstrates that ability and, in addition, an understanding of why the algorithm works, and an ability to write clear mathematics. By many grading standards that I have seen, Answer 1 would be considered perfectly correct. I am proposing that Answer 1 be given very little credit indeed.

Ask questions that require students to decide for themselves what techniques to use and how to fit them together. Here is a question that may seem very easy, but that gave students in our multivariable calculus course a lot of trouble on the final exam:

- *Question:* Consider the plane $2x + y - 5z = 7$ and the line with parametric equation $\vec{r} = \vec{r}_0 + t\vec{u}$. **(a)** Give a value of $\vec{u}$ which makes the line perpendicular to the plane. **(b)** Give a value of $\vec{u}$ which makes the line parallel to the plane. **(c)** Give values for $\vec{r}_0$ and $\vec{u}$ which make the line lie in the plane.

Consider part **(a)** of this question. I suspect that most of the students in this course knew how to find the normal vector to a plane, given its equation, and how to find a parametric equation for a line parallel to a given vector; if they had been given questions which explicitly asked them to perform these tasks, they would have had less trouble. Looking at their answers to part **(a)**, however, I realized that their difficulty was that it was up to them to choose these techniques and put them together. The problem was exacerbated by the fact that the two techniques came from different parts of the course. Many students simply did not know where to start. Their problem was not a lack of facility in using tools, but a lack of judgment about which tools to use, and when to use them.

Test conceptual understanding. I have been surprised in discussions with my colleagues to discover that this principle is not unanimously endorsed. Indeed, it is often used as a defense for teaching calculus as a pure techniques course that nothing more can be expected from students. Another argument that I have heard is that conceptual understanding flows eventually from a rigorous grounding in the 'basics', so that it is not necessary to test it separately. This is not borne out by my experience with the following question.

- *Question*. Suppose that $f(T)$ is the daily cost to heat my house, in dollars, when the outside temperature is T degrees Fahrenheit. **(a)** What are the units of $f'(103)$? What does $f'(103) = 0.87$ mean? What are the practical consequences of this fact? **(b)** If $f(103) = 10.54$ and $f'(103) = 0.87$, approximately what is the cost to heat my house when the outside temperature is $100°$?

The correlation between ability to answer this question satisfactorily and technical ability is weak at best: I have seen many students with excellent algebraic skills who are unable to answer the question at all, and many who are quite weak in algebra who give very good answers. It seems wise to me to test both dimensions, rather than just one.

Avoid Excessive Use of Templates There's not much point in giving an example here: Any question can serve as a template, with sufficient repetition of the type. The traditional calculus course provides many examples, but the different calculus courses that have been developed over the last ten years are just as susceptible to the problem.

The problem with template problems is that students can memorize how to solve them. A template problem on a final exam thus becomes a test of the student's ability to memorize. There are, of course, times when that is exactly what one wants to test (rules of differentiation and integration). But much of the time it leads to absurd results. For example, a standard calculus question is to ask the student to compute $\lim_{x->a} f(x)$, where $f(x)$ comes from a fairly well-defined and limited class of functions (e.g., rational functions). Students will memorize how to answer such questions using some version, perhaps fancier, of the following rules: Try setting $x = a$; if you get $0/0$, cancel powers of $x - a$ and try again. This is good algebra practice, but a poor test of whether students understand the concept of limit. The first question in the next paragraph provides, I think, a better test of that concept.

Ask Students to Reason From Graphical and Numerical Data, in Addition to Reasoning Algebraically Asking students to work with different ways of representing the same object encourages them to come to grips with the underlying concepts and not rely on memorization.

- *Question*. There is a function called the error function which is used by statisticians and denoted by $\mathrm{erf}(x)$.

  (a) Given that
  $$\mathrm{erf}(0) = 0$$
  and
  $$\begin{aligned} \mathrm{erf}(1) &= 0.299793972 \\ \mathrm{erf}(0.1) &= 0.03976165 \\ \mathrm{erf}(0.01) &= 0.00398929, \end{aligned}$$
  estimate $\mathrm{erf}'(0)$, the derivative of erf at $x = 0$. Only give as many decimal places as you feel reasonably sure of, and explain why you gave that many decimal places.

  (b) Given the additional information that
  $$\mathrm{erf}(.001) = 0.000398942,$$
  would you change the answer you gave in **(a)**? Explain.

This question is subject to the following criticism, which is often raised against questions based on numerical data: From a strictly logical point of view, there is nothing that one can say about $\operatorname{erf}'(0)$ using the given information. Nonetheless, it is my contention that a student has demonstrated a good intuitive understanding of the concept of limit if he or she answers part **(a)** by giving some sensible approximation such as $\operatorname{erf}'(0) \approx .40$, then improves it to $\operatorname{erf}'(0) \approx .3989$ for part **(b)**, and explains his or her choices by pointing out when the digits in the first four decimal places of the difference quotient appear to stabilize.

Here are a couple of questions based on graphical reasoning.

- *Question.* Below is the graph of the derivative of a function f; i.e., it is the graph of f'.

  You are told that $f(0) = 1$.

  (a) On what intervals is f increasing? Explain your answer.

  (b) On what intervals is the graph of f concave up? Explain your answer.

  (c) Is there any value $x = a$ other than $x = 0$ in the interval $0 \leq x \leq 2$ where $f(a) = 1$? If not, explain why not, and if so, give the approximate value of a.

You might ask: What is the difference between this question and one where the students are given a formula for f' and then asked the same questions about it? I think that both are fairly good questions. However, the algebraic version of the question is subject to the following problem: Many students have learned a recipe for determining where a function is increasing, decreasing, concave up or concave down. It involves setting formulas equal to zero, solving equations, drawing sign diagrams, taking derivatives, and so on. It is possible to be able to perform this procedure and still not understand that a function is increasing where its derivative is positive. That is, it is possible for a student to be able to answer the algebraic form of the question and be completely helpless when confronted with the graphical version.

Of course, there could also be students who can answer the graphical version, but are unable to answer the algebraic one. That is why I said that graphical and numerical reasoning should be introduced in addition to, not instead of, algebraic reasoning. If you keep approaching things from different points of view, you have a better chance that students will attempt to understand the underlying concept rather than rely on an ever increasing store of memorized procedures.

It is also important to ask questions that require students to translate between different points of view. Here is a simple example.

- *Question.* Below are graphs of $\sin(ax)$ and $\sin(bx)$, where $a > b > 0$. The scale on the axes is the same in both graphs. Which is the graph of $\sin(ax)$, (I) or (II)? Justify your answer.

To answer this question, the student is required to relate the visible difference in periods to the algebraic fact that $a > b > 0$. This requires solving the equations $\sin(ax) = 0$ and $\sin(bx) = 0$, and then giving a geometric interpretation to the algebraic form of the solutions. (One can imagine arguments involving experimentation with a graphing calculator which do not require any algebra; however, such arguments would be weaker and, presumably, worth less in the grading.)

4. . . . Is in the Eating

There are many ingredients in addition to the exams that go to make up a curriculum: the textbook, the method of teaching, and the use of technology, for example. All are important and are rightly the subject of discussion. However, they all represent the good intentions of the teacher, the hopes and plans. Our discussions about the curriculum are so volatile, I suspect, because they are grounded in fantasy and wishful thinking, rather than a realistic picture of what is actually going on inside our students' heads. Good intentions are often unrealized; hopes dashed; plans modified. It is the exam in December or May that reveals what is really in the course, not the book we chose in June or the syllabus we distributed in August.

Whether you are a Sage on the Stage or a Guide on the Side; whether or not you teach the Mean Value Theorem; whether you teach in a classroom full of computers running `Maple`, or ban calculators from your course altogether; it is your exams, and your students' answers, that tell what you have achieved as a teacher. Moreover, it is your exams that tell the students what they really must study. Any course that has been stable for a few years has a reputation, communicated through copies of previous exams and through the experiences of former students. Students pay attention to this reputation; they look at previous exams; they listen to the folklore surrounding the course. Much more than anything said by the Sage or the Guide, it is the course's reputation that determines what students study in the days before the final, and thus determines what knowledge and abilities they ultimately retain.

Teaching or Appearing to Teach: What's the Difference?

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Ask most professors, department chairs, or deans about the quality of teaching at their institution and they will tell you that their faculty are excellent teachers. How do they know? Students tell them so is the most likely answer. Anonymous student evaluations at the end of the course commonly provide the required evidence. And, they will say, very few students complain about teaching. I wonder, however, if there is real evidence that teaching and learning are actually occurring? Without concrete evidence that students are learning, how can there can there be credible evidence that teaching has occurred, excellent or otherwise? To put it more succinctly, you can't have good teaching without good learning. So, the question is "What has actually been learned?" By ignoring this fundamental question and relying solely on student opinion for the definition of teaching quality, administrators, professors, and students become co-conspirators in perpetrating a fraud: This is the tacit agreement, quoted anonymously at the start of Chapter 1 of Krantz's book, between instructor and student to the effect that instructor will pretend to be teaching, the student will pretend to be learning, and both will state that the other is doing a good job. One has an illusion of good teaching without the substance, a very real occurrence when the appearance of acceptable teaching performance is a condition of employment.

There are many ingenious ways in which this fraud is perpetrated. For example, an instructor could offer an "exam review sessions" and "practice exams" that provide students with model questions from which exam questions are superficial modifications. Another approach is to focus on the memorization of basic vocabulary and the acquisition of basic algorithms that can be applied to a small set of stereotype problems. The result? Students are happy, often believing that they have learned something because they did well on the test. They are happy with the instructor who has provided them with what they needed in order to be successful. Therefore, strong teaching evaluations are given. But what has the student actually learned? Quite often, it seems, students have been successfully trained to respond to specific stimuli with specific predictable responses. "Automaticity" and accuracy are the goals, we are told by the "successful teacher". Is it an appropriate college or university goal to create human automata? Especially in this computer era, we must set higher expectations for our students than to become the new intellectual revolution assembly line robots. Alas, many elements of the educational system appear to be designed for this purpose. And students, parents, teachers and administrators, at all levels, believe that this is the goal of education. To be a truly successful teacher, one must confront the issue of what is learned by the student. In the end, this is what really counts.

Possessing a deep understanding of course material, acting in a manner that is respectful of student and oneself, speaking clearly, writing clearly, meeting office hours, etc. are all things that support teaching. But, in fact, one can quite successfully follow the prescriptions of the "how to teach" book without any real teaching ever occurring. One way to address what is missing, I believe, is to pay serious attention to what learning is sought in the course. One should first have a clear understanding of the goals of the course. "learning calculus" is not sufficient. Useful goals are clear, concise statements that are understood by instructors and by students. Most importantly, they must be expressed in a way that both can know whether these goals have been achieved or not. Second, both instructor and student should understand how to achieve these goals. This will avoid the common pitfall of purposeless work that absorbs energy, creates the illusion of learning, but is ultimately ephemeral in intellectual substance. Third, instructor and student must understand the manner in which the achievement of course goals will be measured. Here lies one of the more inviting traps in the world of education. It is quite common, at all levels, to equate the ability to "do mathematics" with "passing the test." I recall a student who finally understood the issue. After a long discussion of a test problem, she said, 'I get it, you want me to understand calculus, and I just want to get an 'A' in the course." Neither are professors immune. Ask a colleague about how a course went and you will usually hear, "Oh, about half the students passed."

A key to addressing this dimension of teaching is to have strong, clear goals. Select materials and strategies that assist the achievement of these goals. Regularly analyze whether or not these strategies are doing so. Use these goals, and only these goals, to measure student achievement in the course. Finally, evaluate your own performance as well as your students' in terms of these goals. Teachers are learners too: continuously learning ways to increase their effectiveness. These are the principal dimensions of teaching that I will discuss in this essay. I will close, however, with a short report on a discussion of the book that took place over two months at Santa Barbara. Participants in the discussion were graduate and undergraduate students who are the staff of UCSB's NSF-supported California Alliance for Minority Participation (CAMP) Achievement Program. They organize and facilitate the work of study groups or provide support of students interested in science, engineering, mathematics, technology based careers. I wish to share their reactions derived from our reading and discussion of the book by way of broadening our perspective on the recommendations found there.

Goals

Why focus attention on goals? They are necessary to define the nature and extent of the success. While they are sometimes informal or part of the institutional cultural, they are most useful when they are articulated in a form understandable by new faculty members and students. Formulating these goals is one of the most important responsibilities of a faculty. It seems, this is a responsibility that is frequently avoided or trivialized. Why? It is work that takes time away from students, research, and other scholarly activities. And it is very difficult. During a recent meeting a colleague said, "If we have to agree on the goals for this course, we will never get anywhere. We have as much chance as proving that God exists!" We never did agree on goals. Some argued that they were unnecessary, others said we already knew what they were, and others, disagreeing, that they were essential in order to evaluate proposed texts and syllabi. There was never a resolution of this issue. Nevertheless, a text and syllabus were adopted. In such circumstances it is little surprise that, for example, the course failure rate varies between 10 percent and 70 percent according to who is the instructor. As far as I can determine, these events reflect profoundly different courses despite sharing the same name, text, and syllabus. Individual instructors formulate their own goals, as they must, in such circumstances.

So, what are good useful course goals? Ideally, each course offered by your department has a few clear academic goals but often this in not the case. Goals are not the same as list of topics, chapters, or sections to be covered. Goals must make sense to you and your students. All of your must be able to know whether or not they have been attained? To give a first example, let me return to the earlier meeting discussion of calculus texts and syllabi. A colleague said, "I know one of Millett's goals, he wants his students to be able to solve problems they have never seen before." And it's true, that is one goal I have for many courses. It is not a goal shared by my colleagues, neither explicitly nor implicitly. But I think that it is a good an example of a goal. Easy to understand and advancing a key component of a university education, especially in mathematics. Other beginning calculus course goals are: "Understand the relationship between math and the 'real world'." "Assume personal responsibility for learning." "Strengthen critical thinking and problem solving skills." "Acquire a concrete understanding of fundamental concepts." "Develop basis calculus skills." "Increase the number of students making steady progress in math and science programs." I have twenty four such goals, many of which naturally expand into a more detail description. Those associated to the "basic calculus" skills will seem familiar to those who have used the Harvard Consortium curricula. I share these goals with my students and point to them, where appropriate, in explaining assignments or course structure. In a second year introduction to advanced mathematics course, goals included: "Be able to read and explain mathematics." "Be able to distinguish between a complete and correct proof and a faulty argument." "Be able to write and explain simple proofs." And, finally, in a linear algebra course, a very challenging goal was: "Be able to extend concepts, methods, and results from finite dimensional vector spaces to Banach or Hilbert spaces." These are the kind of goals I use in contrast to "Calculate the derivatives and integrals of polynomial functions," or "Create a truth table for simple propositions." or "Be able to state, prove the Hahn-Banach theorem." At all levels, students often believe that they need only memorize an algorithm or body of material in order to be successful. They do this in much the same way that they would memorize a poem or learn to juggle three balls. While both may be good things to learn, they are not good models learning mathematics in a form that will be useful beyond a grade in a specific course. And becoming able to do so is, after all, what we are all about. Our goals should enable this to occur.

In addition to the challenge of developing a faculty consensus on goals, there are other dimensions that should be considered. Are the goals understandable by students as well as instructors? Are the goals appropriate for the students enrolling in your course? Students will need to have had the appropriate preparation to have a fair opportunity to achieve the goals. Are there appropriate and sufficient resources to goals? For example, is the size of the class appropriate? Does the textbook promote or inhibit student success in achieving these goals? As I mentioned earlier, in my calculus courses one of the goals is to develop an ability to solve unfamiliar problems. To quote a former student, "To learn what to do when you don't know what to do." Textbooks organized around a collection of standard problem types and training in the use of standard algorithms have proven to be an insurmountable barrier to achieving this goal. Since many of the traditional texts have this structure, they would not qualify for adoption in a course that I would teach. The reason is that most of my students come directly from high school experiences that have convinced them that this is what math is about. It is extremely difficult to be open up their thinking to alternative views on this matter. Thus, if this is one of the goals you will not be able to use such texts and, if you are required to use such a text, this would be an unachievable and, therefore, inappropriate course goal.

Another example in which the specific goals are critical would be determining the array of teaching strategies to be employed in the course. The abilities and experience of teaching assistants or homework readers also have significance for the choice of goals and the support or training needed to enable them to help in the achievement of the goals. In large public universities many graduate students have not been educated in this country. They, and many of those who have, find it very difficult to stimulate, organize, or guide small group student discussions of assigned problems. In some cases, I have failed to get them to even attempt to do so. They have become obstacles to achieving the goals and, in some cases, have undermined my best efforts with certain groups of students. The sad result is that rather than helping these students they have caused some to achieve a level of performance needed to continue in mathematics. These students have had to seek an instructor who goals were compatible with their own thinking and retake the course. For example, if one of my goals is to increase the quality and quantity of mathematical communication and the strategy of small group discussions is quite effective in providing opportunity to practice explaining calculations or solutions of problems. As I do not give full credit for work without full explanations, students who are unable or unwilling to provide them will not be fully successful in the course.

Strategies

Goals without strategies to attain them are useless. Strategies depend not only on the course goals but, as do the goals, depend upon the circumstances, the preparation and attitude of the assistants and the students in the class. Many experienced instructors have a large and varied array of instructional approaches available to them. Novices may be constrained to the most familiar, those they have observed as students and which they feel most able to imitate. What ever the choice or choices, the selection is determined in relationship to the goals. If the goals are superficial, then superficial strategies are all that are required. If, however, you have chosen a more challenging array of goals, then you may need to improvise, modify, create, experiment, etc. over a period of time. For everyone, an ongoing evaluation of the observed results needs to take place routinely in order to determine to whether learning is actually occurring and course goals are being accomplished. The real teaching professional continuously elaborates upon previous approaches, develops new ones, and measures progress in order to make "mid-course" modifications.

In public universities we are often called upon to teach classes of quite different sizes and purposes. The strategies will change even though a goal may remain the same. For example, the development of stronger mathematical communication abilities and the acquisition of self-confidence and personal responsibility for learning are among my beginning calculus course goals. As a consequence, even in classes of 150 students, I frequently use a "small group activity" strategy that requires students to interact mathematically with classmates and asks them to share, even in the large class setting, their results and questions with the whole class. Having a seventy-five minute class requires that I monitor the extent of engagement of the students with the material I am trying to communicate. They often "fade" after about fifteen to twenty minutes. At this point I might choose to introduce a new topic or idea by giving them a "surprise group quiz". Of course my purpose is NOT to test them on material I have not yet discussed, but they should have read, but to get them thinking about the problem or topic, to change the energy level, to redirect and re-engage their thinking, and to prepare them for what I would like to talk about next. This, I find, can work surprisingly well. I give them about five to ten minutes, ask them to share their conclusions or solutions. Often they use the board or overhead. I ask for alternative solutions, for different answers, In general, I work hard to express appreciation for every student's contribution. When there are competing answers or solutions, I ask that they all be explored in a supportive and civil manner that does not compromise on our goal seeking a correct and understandable answer. This can work VERY well and usually provides me with all the material I need to introduce and discuss the topic I had in mind. But you need to be "on your toes" to fully take advantage of this approach. I once asked a class to work on a problem leading to multiple integrals. Rather than proceeding as I had expected, one group of students took an approach leading to the Lebesgue integral while others took the expected path. The resulting discussion required about an hour, brought out some misunderstandings about functions of two variables, lead to a comparison of numerical estimations, and to a comparison between the two approaches in terms of an ability to do symbolic calculations. I was exhausted by the end of the class. Years later students who participated in the discussion would bring it up in conversations about their experience in my class. This only happened once in recent times, but it is an excellent illustration of the sort of thing I am seeking from my students and how one can go about trying to make it happen.

In another experiment, I have been trying a strategy to increase the intensity and effectiveness of interactions with students and to improve the level of student performance on homework. I have converted my office hours into a three hour session in the "math lab" during which each student is required to personally present her or his homework each week. From their papers I read the name of each student and greet them. This helps me to know them better and gives the class of 150 students a smaller feel. I then scan a selected problem or two and give the student my quick assessment of the quality of the submitted work. If it is up to "standard" I tell them so and wish them a good day. If it is not, I tell the student where there appear to be deficiencies and offer them a chance to do an immediate revision, with the assistance of the graduate student staff working in the lab. In smaller classes, I have asked the students to come to my office rather than the lab. I have them sign up for a specific Friday interview time, about 6-7 minutes each for a class of 45 students, either with me or a graduate assistant. We receive their homework, scan a problem, ask them a follow-up question or two, and, if time allows, discuss questions that they might have about the course material. If the homework is not up to the course standard I offer them the option of continuing to work on it over the weekend to attempt to improve the work. In some cases I have refused to accept the work because it was so poorly done and have insisted that the student make a brief appointment with me the next week to review their progress. In short, I would only accept work that appeared to be of passing quality.

I mention these examples to expand the range of strategies that might be considered and because they represent strategies that are new to me, even after teaching for about thirty-five years. While they may be familiar or time worn to some, I find myself continuing to evaluate the degree to which they help students enrolled in my classes reach a higher level of accomplishment. This must be directly tied to the course goals. It is worth the effort? There are at least two differing views or priorities that seem to determine the answer. For those willing to devote roughly 30 hours per week to teaching, as I have tried to do, the increased level and rate of success and greater interaction with students supports an affirmative response. For those needing to minimize the number of hours required to be evaluated as a good teacher, strategies of this sort are not the best. I work on a base of 60 hours per week with at least 30 devoted to research, scholarly and administrative work and the remainder devoted to teaching. In a research university there is an intersection of the two due to the time spent supervising graduate research.

Success or Failure?

The bottom line is you haven't been teaching if your students haven't been learning! So both you and your students need to know whether the course goals have been achieved or not and to what degree. You will need to be able to determine grades or award credit for the course. But also, you must consider future changes in goals, strategies, or methods of assessment based on your experience in each course. What sorts of things should be considered in the first step, the evaluation of student work?

There are many approaches to grading, ranging from the implementation of a uniform standard examination system found in multi-section courses at some large public universities through complete individual instructor autonomy. In some cases a “grading curve” is imposed on the instructor while in other institutions a “standards based” approach is employed. I am, and have been for a most of my career, a believer in the standards approach. This came about many years ago when I discovered great differences in performance between an 8 AM calculus class and an 11 AM class that I was teaching. As these differences were consistent over a period of several terms, I have adopted a more objective approach to awarding grades rather than using a “curve” for each of the classes. I needed to develop, articulate, and teach according to a set of appropriate course goals. And I needed to evaluate student work based upon these goals. With the advent of common final exams and an ability to compare performance and grades across sections, I have had to raise my average grade a bit in order to insure equity across instructors. While many do not subscribe to this quantitative approach, I strongly recommend it as a consistent and fair method. Instructors will need to look closely at this question before beginning the course in order to insure that students are informed on exactly how grades will be determined.

How does one attempt to measure student achievement? I mentioned above my experiments with weekly short interviews. We gave credit (the amount was the equivalent of a graded homework problem) if a student can explain a problem and/or its solution at a level appropriate to the course. If relevant work was not submitted for the chosen problem, the grade was 0 according to a scale that I will discuss later. The underlying principal I have adopted with respect to the interview questions and, indeed, any work asked of students is: “If it’s important to the course goals, we will ask you and, if not, we won’t.” For example, students are asked to explain their solutions to assigned homework problems. They can also be asked to discuss questions that have not been assigned or material from the reading assignment in the text.

With respect to examinations, I believe that there should be “no surprises.” For my students, not to see an problem for the first time on an exam would be surprising since one goal is developing an ability to solve unfamiliar problems. If it is important in the course, it should be part of the evaluation. If not, it has no place there. No “trick questions” is another maxim I try to follow. Perhaps the most radical thing I do connected with the adoption of goals is to evaluate student performance on against a five point scale as follows. Each unit of work, interview, homework problem, or exam problem (sometimes portion of a multi part problem) is awarded credit according to the scale: 0 points: no progress or relevant information; 1 point: some visible progress that could lead to a solution or correct response; 2 points: significant progress, many major elements present, a partial explanation or proof; 3 points: essentially complete and correct solution but with only minor gaps, errors, or lack of explanation; 4 points: fully correct and complete solution including explanation or proof as appropriate. This evaluation scheme has been difficult for graduate assistants, undergraduate readers, and my students to adopt and to understand. In addition, it has been difficult to use in making comparisons with the credit awarded by other instructors (but their lack of specific measurable goals makes this difficult under any circumstances). Alas, many students confuse the numbers with grade points and while there is a connection it has not been easy for them to internalize after years of experience with another system. I actually believe this has been helpful in focusing attention on what they have actually accomplished. And there is no pleading for "partial credit". Even if they have presented pages of material, if there is nothing that could lead to a solution the mark is 0. The discussions that I have had with both students, graders, and assistants have helped us all keep a focus on the goal fully complete and correct solutions, with full explanations and proofs in contrast to an accumulation of points based on "partial credit" as a strategy for success. Since I allow only integral grades decisions have to be made as to whether on not work meets the standard for a 4. These discussions can lead to the creation of a rubric and benchmark solutions and can be the focus of a rich "professional development" opportunity as you interact with graduate students or undergraduate student readers. What does a "4" really look like? How can two solutions that look very different both merit the same mark? Isn't one "better" than the other? The development of model problems and exemplars to illustrate what each of the marks means can also be a helpful activity. By the way, current translation to letter grades makes 3.3 and above an A (one student received a 3.9 in one of my courses this term) and below 2.0 is not passing (for my institution, this means a "C−" or lower).

What has been the results of these experiments? Here is one point where the goals are critical. I could and have responded to such questions from colleagues and others by recounting anecdotes. For some purposes, this might be sufficient. But, at least for myself, I need an approach that will help me decide on continuing, modifying, or abandoning a particular strategy in favor of more productive ones. I need to weigh the costs against the benefits, for my students, my assistants, and myself. Anecdotes are not adequate for this task. Neither are the "feelings" that we often cite, even those based on thirty years of teaching experience. An approach that is less subjective and influenced by a desire for confirmation of prejudices, one that is as objective and analytical approach as one can easily construct is required. One must recognize and respect the limitations of drawing conclusions, especially with respect to ones own teaching, from the course data. But I know of no other choice. I try to bring the same quality of thinking to my teaching as I bring to my research, and the same skepticism and demand for "proof". At least proof to the extent that it is feasible in this context.

One very useful task that I have done in the past is an item by item analysis of the final exam work do determine whether or not the problems appeared to capture the information that was being sought. This has proved very useful while teaching first year calculus and has lead to some important changes in the types of problems that I have asked as well as leading to shifts in emphasis during the course. What then have I observed in my introduction to proof course? An analysis of final exams and homework has high lighted the persistent problem of students to attempting to memorize certain proofs, either as found in the text or gotten from class discussion sessions or lectures. When grading 45 of them, one recognizes them their similarity, identical notation across students and a loss of precision or focus. In addition, one finds fragments of arguments and a lack of completion or resolution. This sense has been confirmed by students in our interviews. This is a common problem. One faced by all instructors in this course and is a frequent topic electronic discussion groups concerned with teaching mathematics at a university level. I have tried some of the approaches suggested there, such as asking students to explain a question or problem prior to attempting a solution. This has been quite helpful. Under stress, however, my students seem to fall back on the memorization habits learned at a younger age. They wish to depend on memorization and pattern problems or algorithms, a successful method in high school and earlier. Mostly, these students do not yet have confidence in their own intelligence or ability to think. While I wish I could be certain that I have made progress with my students, and I know I have with a few, I am not satisfied with the result.

Parenthetically, this specific problem is a major reason for my interest in issues of K–12 mathematics education. It is also the reason I am quite dismayed by many of the recent proposals to implement a mathematics curricula promotes the acquisition of facts, standard algorithms, automaticity, and abandons efforts to develop a stronger capacity to think and communicate mathematically, to solve unfamiliar problems, to successfully use mathematics in a wide range of circumstances. Mathematics often appears to mean pages of exercises, multiple choice tests and has nothing at all to do with what should be going on "between the ears". Selecting students who are successful at this sort of anti- intellectual exercise for admission to a university where, if we stand for something at all, it is valuing curiosity, creativity, individual thinking, and puzzling things out makes no sense. It seems to be a process designed to optimize the frustration of professors and their students. Addressing this is the real remedial educational required at the university.

What is the bottom line? Less than 10% of the students received failing grades in my class compared to the historical average of 32%. A review of class records shows that the failures were precisely the students who did not present themselves for weekly interviews. Although some students complained about the level of effort required of them, many appreciated the opportunity to receive weekly information on their performance. Approximately ten students dropped the course after the first week or two and others may have avoided enrolling due the intensity of effort required. This an important factor that may have lead to a lower failure rate as I have observed the same behavior in calculus classes. There were 26% A's compared with the historical average of 12%. In order to insure that my standards were similar to those employed by other instructors I reviewed the graded exams from two other courses and talked with several more instructors. In some cases I used exam problems structurally parallel to theirs. I even compared grading standards to try hold to the same or better expectation for performance. These grades are sufficiently different from the historical experience that, if I should ever be asked to teach the course again, I will want to look again closely at the standards that I employed for this course.

I believe that these strategies have impacted several important problems. First, the number of students enrolled in the course who are truly engaged in a serious attempt to pass the class been significantly increased and the failure rate reduced. I believe the pressure of meeting once a week and being confronted, in as sensitive and as direct a fashion as possible, is very much appreciated by most of the students, despite their nervousness. Second, I and my assistants have a much richer and more immediate feedback on how well the students are doing and can make timely adjustments to the course to respond the needs or to take advantage of opportunities.

Student Perspectives

At the University of California, Santa Barbara, I am the Regional Director of the California Alliance for Minority Participation (CAMP) and also coordinate the associated Achievement Program Academic Workshops. These workshops are affiliated with critical barrier courses taken by students pursuing degrees in science, engineering, or mathematics. They are organized and lead by outstanding graduate and undergraduate student staff recruited and trained for this work. During the Spring Quarter these staff members[1] participated in weekly staff meetings as part of their own leadership development and professional growth. During these meetings we discussed the second draft of *How to Teach Mathematics* by Steven Krantz as a means to stimulate thinking about how we could improve the effectiveness of our workshops. In this concluding section, I will share some of the positions developed during these discussions.

First, they were impressed and encouraged by the discussion of changes in opinion of the author and reports of other professors concerned with improving the quality of college and university teaching. The fact that professors might continue to work at becoming "better" teachers was new to many of the participants. The conversations frequently returned to the topic of respect. It arose in several forms: a professor's respect for him or herself, for colleagues and for students, and a student's respect for the professor, for fellow students, and for themselves. For example, as future teachers and current leaders of workshops, there was an appreciation of the issue of self-respect and its significance in terms of their own preparation and development of the skills needed to be successful leaders. The importance of instructors treating students with civility, courtesy, and respect arose in almost every discussion. They felt that often professors did not respect their students. Among the other ways in which this lack of respect was manifested included not being prepared for the class or not having mastered the material, not arriving on time, not ending the class on time, and by not insuring a productive class environment by allowing students to wander in and out of class, to talk in class, or to otherwise disrupt the class. The fact that this issue arose so frequently in the book reflects its importance to Krantz and to them.

While Krantz tells the reader that his book is not a recipe, there are many structural ways in which the book appears to be just that. The suggestions, it was felt, were helpful for all instructors but especially for those not familiar with the K-16 educational culture in the United States. At UCSB we have not hired

[1]Spring Quarter 1998 Achievement Program staff are graduate students Doli Bambhania, Kathi Crow, Ana Garza, Nancy Heinschel, Michael Saclolo, Becca Thomases, and Jeremiah Thompson and undergraduate students Gladis Aispuro, Maria G. Arteaga, Carla Billings, Roxana Cervantes, Hozby Galindo, Analilia Garcia, Hector Garcia, Nicolas Hernandez, Elizabeth Hutchins, Mason Inman, Katrina Jimenez, Christina Luna, Patrick Murphy, Erica Ocampo, Manuel Salcido, SuGen Shin, Shannon Shoup, Edgar Torres, and Ahmad Yamato.

a mathematician who has received an undergraduate education in this country in more than a decade. While sympathetic to the challenges of teaching in a foreign language and culture, the students were concerned that instructors provided them with high quality instruction. Concerns with language and with respect for women and persons other national origins where the principal ones mentioned.

The question of preparation provoked a rich discussion. Graduate students, especially, felt that the amount of time that was required for new teachers was undervalued. Furthermore, one can not really "over-prepare". Rather, one could prepare badly. One analogy is that of the actress or actor who has performed the same role in a play hundreds of times. The lines are not the issue. The challenges are to have a deep understanding of the role, to make character "come alive" time after time, to directly engage each audience, and to "fill the room with your presence," even while fighting a flu bug or distracted by personal problems. For professors, making learning living intellectually engaging experience, sustaining interest and progress over the period of weeks and months at a high level, and stimulating a search for new and deeper meaning are all critical elements of the teaching craft.

The uses of various pedagogical methods provoked an energetic discussion as well. For example, while some instructors use transparencies or computer displays effectively while others fall into the trap of merely displaying a series of images too rapidly and without sufficient impact. Their use can be a problem for students. Don't "read the book to the class." Students should be reading it themselves. Repeating material in the book undermines the value of the class meeting. Much more should be expected from the instructor by way of establishing priorities and making choices of material! Lectures by charismatic speakers and small seminar discussions which are "content-free" are a problem. Some instructors do not appear to be genuinely concerned with whether or not their students actually learn anything of substance. "I won't disrupt your life if you don't disturb mine," appears to be the guiding principle of some. Actual engagement or interaction between professor and student is to be avoided. In contrast, students argue that the focus should be on what is good for the students, not what is good for the professor. Maximize learning, not minimize disruption.

Don't "read the book to the class." Students should be reading it themselves. Repeating material in the book undermines the value of the class meeting. Much more should be expected from the instructor by way of establishing priorities and making choices of material!

Conclusion

I have very much appreciated this invitation to spend some time reading and reflecting on the second edition of this book. I believe that it will be a useful resource for persons wishing to improve the teaching of mathematics at the college and university levels. My student colleagues certainly reacted very favorably to the material in this edition and the discussions helped us all better understand some important elements. The CAMP staff discussions had a tendency to revisit certain key issues following the course of the reading. At one point, the recognition of recurring themes became the focus. For example, the concern for respect recurs often. Because it is so important to students and is seen to a serious problem that inhibits their learning, the CAMP staff were quite encouraged by the treatment it receives.

In one of the concluding discussions, I told of my perspective on the book and concerns I wished to address in this essay. These did not receive the same interest as did the topics in the book. Those issues are much closer to the immediate experiences of the students and are more familiar to them. As such, they represent an good collection starting places. Indeed, a more appropriate title might have indicated that this is really only an introduction to teaching mathematics. Excellent for novices, it does not address adequately some of the issues that I have tried to introduce in this essay. There is, I believe, very much more to being a successful mathematics teacher than is presented in Krantz's personal perspective. It is too easy to ignore the fact that unless our students are learning, we are not actually teaching. To ignore the implications of this undermines the fundamental and historical role of mathematics departments in most universities. How do we know our students are learning? What is it, exactly, that they are to be learning? What are the goals of our courses and of our teaching? Are we succeeding or failing?

As challenging as it might seem, we need to be able to answer these questions for ourselves, for our students, and for out institutions. It needs to be a major concern for persons starting out teaching careers. And it needs to be a continuing concern, especially for those of us who have been teaching for decades May our teaching efforts be described more favorably than with Macbeth's words, "Life's but a walking shadow, a poor player that struts and frets upon the stage and then is heard no more: it is a tale told by an idiot, full of sound and fury, signifying nothing."[2]

[2]William Shakespeare, *Macbeth*, Act V, Scene 5

Why (and How) I Teach without Long Lectures

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1. Why I Gave Up Long Lectures

I could have benefited greatly from Steven Krantz's tips in 1962 when I taught my first class. In fact, I can see that over the years my lecturing style and techniques evolved to be remarkably similar to those Steven Krantz (SK) suggests. I was a very popular lecturer and recently won an MAA sectional award for distinguished teaching based in no small part on the lecture courses I gave at Illinois between 1968 and 1988. But for the last ten years, I have completely abandoned the long lecture method.

My last lecture effort was calculus in 1988. I thought I did a bang-up job, but the students did not respond with work anywhere near the level I was used to and have become used to after I gave up on introductory lectures—despite the fact that I had been giving the lectures largely in harmony with SK's recommendations.

Simply put, today's students do not get much out of long lectures, no matter how well they are constructed. The material comes too fast and does not sink in well. The students of the past responded by becoming quiet scribes. Today's students demand more action and accountability. That's why many students cut class and even when they come they often ask hostile questions such as "What's this stuff good for?" They do not read their texts. Some students even disrupt lectures. And as SK notes, many professors ask the questions

- Why won't my students talk to me?
- Why is class attendance so poor?
- Why won't students do their homework?
- Why do they perform so poorly on exams?

And then they shrug it off saying to themselves: "If only I had taught at Harvard things would be different. I would have bright and eager students." or "Students these days are impossible."

It is the lecture method of teaching that is impossible—the method of teaching via long lectures is crumbling under its own weight. This is true not just in mathematics. Across the University of Illinois, there is a major controversy about whether professional note takers may take notes and sell them to students who would rather not attend lectures. One of the first to note that the lecture system needed to be replaced was Ralph Boas in 1980: "As a means of instruction, lectures ought to have become obsolete when the printing press was invented. We had a second chance when the Xerox machine was invented, but we muffed it." Many math instructors are trying to teach today's students using only yesterdays tools and approaches. And neither the instructors nor the students pleased with the results.

Introductory lectures are not (and probably never have been) a particularly effective vehicle for introducing students to new material. A few strategically timed and strategically placed short follow-up lectures (sound bites) can be very effective. But the problem with introductory lectures is that they are full of words that have not yet taken on meaning and full of answers to questions not yet asked by the students. A further problem is that many lecturers fall into the trap of believing that their job is to think for the students. This effectively shunts the students to the sidelines—making them into mere scribes who verify in the homework and tests the math truths promulgated by the lecturer. As Bill Thurston put it: "We go through the motions of saying for the record what the students 'ought' to learn while students grapple with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material 'covered' in the course, and then grading the homework and tests on a scale that requires little understanding. We assume the problem is with students rather than communication: that the students either don't have what it takes, or else just don't care. Outsiders are amazed at this phenomenon, but within the mathematical community, we dismiss it with shrugs."

In summary, I do not disagree with SK's approach to lectures, as he gives some great advice, which I used to follow as well. However, I do question the necessity, importance, and educational quality of lectures as a method for students to learn mathematics.

2. What I Replaced Lectures With

Another piece of wisdom from Ralph Boas :"Suppose you want to teach the 'cat' concept to a very young child. Do you explain that a cat is a relatively small, primarily carnivorous mammal with retractile claws, a distinctive sonic output, etc.: I'll bet not. You probably show the the kid a lot of different cats saying 'kitty' each time until it gets the idea. To put it more generally, generalizations are best made by abstraction from experience."

Today my calculus, differential equations and linear algebra students get the experience they need through `Mathematica`-based courseware written by Bill Davis, Horacio Porta and me. The basic ideas are laid out in interactive `Mathematica` Notebooks in which new issues arise visually through interactive computer graphics. With this courseware, limitless examples are possible almost instantly. If the student doesn't get the point right away, then the student can rerun with a new example of the student's own choosing. They can use the courseware to touch and see the math "kitty" as many times as they want to. They see for themselves what the issues are before the words go on and generalizations are made. One of our favorite techniques is to give a revealing plot and ask the students to write up a description of what they are seeing and to explain why they see it. In these courses, conceptual questions are the rule and students answer them. Contrast this with the typical student problems assigned in traditionally taught mathematics courses.

Here is the story behind the evolution of our courseware and the way it is used: In 1988-90, when Horacio Porta, Bill Davis and I were developing the original version of the computer-based course Calculus&`Mathematica`, Porta and I offered regular introductory lectures at Illinois. We noticed poor attendance and asked the students why. The students uniformly replied: "We don't need them. We can get what we need from the computer courseware when we need it. What we do want is a followup discussion from time to time." We followed their advice and have never seen the need to go back. Our students taught us how to teach. Over the years, almost all teaching of Calculus&`Mathematica` (and sister courses DiffEq&`Mathematica` and Matrices, Geometry&`Mathematica`) has evolved to this model (sometimes known as Studio learning, a term coined by Joe Ecker for his `Maple`-based calculus course): All the student problems are freshly written with the idea of engaging the student's interest. Assignments are made on Thursday. Students work on each assignment for one week. One day before the assignments is due, a classroom session is held to discuss what the week's work was all about. Students come armed with questions and if they don't fill up the whole hour, then the instructor gives a several pointed mini-lecture addressing points the students should have picked up during the last week. All other class meetings are in the computer lab with the instructor answering student questions as they arise—at the ultimate teachable moment. This lab interaction between teacher and student (which is sometimes done via e-mail) is very important. No longer are the students the professor's audience; students are the professor's apprentices.

The students' weekly assignments count for at least half their semester grades. Because there are no other lectures, the whole course consists of student work. In this model , it is what the students do that is important rather than what the professor says and how professor says it. Still the influence of the instructor is pervasive and the course ends up satisfying Gary Jensen's and Meyer Jerison's criteria: Setting pace, teaching students to read, and fully engaging the student in the learning process.

This learning model cannot be accomplished with traditional printed texts and traditional lectures and lends itself rather well (but not perfectly) to Internet distance education. NetMath centers offering via the Internet calculus, differential equations and linear algebra courses for university credit supported by live mentors have formed at several universities and colleges. Here is a reaction from a high school teacher who sponsors NetMath Calculus&`Mathematica` in Alaska: "Jessica has really enjoyed the course, and her father, a veteran of traditional calculus courses, is very impressed with the understanding of the mathematics that this method imparts. He has done all the problems and loved it. There have been some loud arguments—most of which she has won."

3. Content Issues

The trouble with the lecture system is compounded by the fact that our undergraduate courses, for the most part, have been frozen in the past and have become unable to adjust to modern demands. Undergraduate mathematics courses today are nearly indistinguishable from the undergraduate courses I took in 1960. Peter Lax put it this way in 1988: "The syllabus has remained stationary, and modern points of view, especially those having to do with the roles of applications and computing are poorly represented . . . " When I look over mathematics undergraduate courses during this century, I see a smooth evolution of new ideas and better mathematics through the period 1900–1960. Topics of limited interest such as haversines, common logarithms, Hoerner's method, latus recta, involutes, evolutes, Descartes's rule of signs all had their time in the sun but were de-emphasized in favor of more important topics. And then the content became frozen. There is a whole list of 20^{th} Century topics that have been by and large rejected in today's mathematics classroom. A short list: The error function, singular value decomposition for matrices, unit step functions and their "derivatives", the Dirac delta functions in differential equations, using the computer to plot numerically solutions of differential equations, Fast Fourier Transforms, wavelets. There is plenty of what Peter Lax calls "inert material" in most of our current mathematics courses. It's time to get rid of it and open the door to some fresh, important material.

My bet is that the underlying cause of this is our current fanaticism about having one-size-fits-all uniform texts chosen by central committees who often lack the expertise to make significant changes. They just go on tinkering with what was done the year before. It seems the central committees do not trust the initiatives of individual faculty members, so they shackle them with obsolete material. Publishers respond in kind. And the publishers stay away from texts for modern courses because new, modern, original texts are unlikely to sell well. This is the reason that most well-selling traditional calculus texts are clones of George Thomas's calculus course of the 1950s.

The trend is for engineering, biology and science departments to begin teaching the mathematics their students need. Mechanical engineering departments are teaching lots of advanced calculus and differential equations. Electrical engineering departments are teaching lots of probability and complex variables. According to a source inside the Stanford University Computer Science Department, they have decided:

a. They'd like their students to have more math.

b. But not the kind of math that's coming from the math department.

c. It's never going to come from the math department.

d. They'll start doing it themselves.

No wonder Sol Garfunkel and Gail S. Young wrote: "Our profession is in desperate trouble—immediate and present danger. The absolute numbers and the trends are clear. If something is not done soon, we will see mathematics department faculties decimated and an already dismal job market completely collapse. Simply put, we are losing our students." Are mathematics professors and departments in extreme denial? I wish SK had dealt with these serious issues.

4. Specific Remarks about SK's Revision

SK: "We do not want our students to learn to push buttons. We want them to think critically and analytically. It has been argued that `Mathematica` and similar software to help students interact dynamically and visually with the graphics: Vary the value of a in the equation $y = ax^2 + bx + c$ and watch how the graph changes. That is not what I want my students to learn. I want them to understand that, for large values of x, the coefficient of a is the most important of the three coefficients. And changing its value affects the first and second derivatives in a certain way. And, in turn, these changes affect the qualitative behavior of the graph in a predictable fashion. AFTER [SK's emphasis] these precepts are mastered, the student may have some fun verifying them with computer graphics. BUT NOT BEFORE [my emphasis]."

Reaction: Why not before? Is this a moral issue? Certainly this is not an educational issue. Students (and research mathematicians) learn lots from examples. Here is my version: Have the students play with graphics, first varying a and coming up with a conjecture of what the influence of a is. Then ask them to explain why their response is correct. Then ask them to explain how changing the value of a affects the first and second derivatives in a certain way and how this is reflected in corresponding plots. In this way the students engage completely in Saunders MacLane's sequence for the understanding of mathematics: "intuition-trial-error-speculation-conjecture-proof." I don't care how many buttons students press; if it helps them to think critically and analytically, I'm all for it.

SK: In a programmed learning environment, whether the interface is a PC or with `Mathematica` Notebooks or with a MAC, the students cannot ask questions.

Reaction: Disagree. Students at computers can and do ask questions—lots of them.

SK: "What sense does it make to have a mathematics classroom, with a computer before each student, and the instructor delivering command to the students? ... People need to perform laboratory activities in their own time at their own pace."

Reaction: I agree thoroughly. A teacher-centered computer lab is absurd. The professor has to learn to relinquish total control when the students are in the lab.

SK: "Most of us were trained with the idea that the whole point of mathematics is to understand precisely why things work. To make the point more strongly, this attitude is what sets us apart from laboratory scientists."

Reaction: I agree and disagree. Throughout our courseware are two recurring themes:

1. One of the goals of mathematics is to explain why things work out the way they do.

2. There are no accidents in mathematics!

On the other hand, good mathematical research has always been a laboratory science. Top quality mathematical research invariably feeds off examples and special cases. With the computer, students can get lots of examples and begin to formulate what mathematical truth might be then go on to explain why—thereby engaging in the whole mathematical process. Students in lecture classes miss out on this opportunity. The lecturer handles all of this for them.

SK: "One of the highest and best uses of computer in mathematics instruction is as the basis for laboratory work."

Reaction: I agree thoroughly. But I hasten to add that not all laboratory work is good. Weekly lab sections tacked onto an otherwise traditional course are of dubious value. The way the calculator is used in high schools to prepare for the AP exam is far from optimal.

SK: "Using the quadratic formula is easy. Analyzing a word problem is hard. A person who cannot do the first will probably not be able to do the second—with or without the aid of a machine."

Reaction: Disagree. One does not follow from the other, as many students in Calculus&`Mathematica` "BioCalc" sections at Illinois have proved.

SK: "If you choose a poor text, you will have to pay for it through the semester."

Reaction: Too many professors are not allowed to choose their texts. And how many really good texts are there out there?

SK: "...the respect a teacher must show his audience."

Reaction: A really good point. But I am uncomfortable with the characterization of students as audience. Audiences are usually faceless and rarely participate; they just watch. The view of students as audience is one of the major defects of the lecture method.

5. Suggestions for Further Reading

Here are some sources for those who want to re-examine their ideas about teaching of mathematics:

a) Ralph Boas's article "Can We Make Mathematics More Intelligible?" (*Amer. Math. Monthly* 88 (1981), 727–731) is a provocative short complement and counterpoint to SK's revision. Samples are included in the text above. This and a number of Boas's other essays were reprinted in the book "Lion Hunting & Other Mathematical Pursuits" (MAA,1995). Enlightening reading!

b) Gian-Carlo Rota's book, *Indiscrete Thoughts* (Birkhäuser,1997). Rota raises the right issues in the way only Rota can. A sample: "One must guard ... against confusing the presentation of mathematics with the content of mathematics. An axiomatic presentation differs from the fact that is being presented as medicine differs from food. ... Understanding mathematics means being able to forget the medicine and enjoy the food." If you read this book, you will not forget the experience!

c) Reuben Hersch's book, *What is Mathematics Really?*, (Oxford, 1997) A sample: "A philosophy of mathematics that obscures the teachability of mathematics is unacceptable."

d) H. Poincaré's books *Science and Hypothesis*, *The Value of Science*, and *Science and Method: Some Running Themes*: the value of intuition, arguing against reduction of mathematics to algebra à la Weierstrass; verification is not enough. Poincaré was one of the first to point out that mathematics teaching is not all it could be.

e) Henri Lebesgue's essays, *Measure and Integral* (Kenneth O. May editor, Holden-Day 1964). This books consists of pedagogical essays written by Henri Lebesgue and assembled by Kenneth O. May. The essays are very heavy on pedagogical and mathematical content with a passionate plea for better acceptance of decimal numbers in the mathematics classroom. Two samples: "There is a real hypocrisy, quite frequent in the teaching of mathematics. The teacher takes verbal precautions, which are valid in the sense he gives them, but that the students most assuredly will not understand the same way."

And:

"Unfortunately competitive examinations often encourage [an educational] deception. The teachers must train their students to answer little fragmentary questions quite well, and they give them model answers that are often veritable masterpieces and that leave no room for criticism. To achieve this, the teachers isolate each question from the whole of mathematics and create for this question alone a perfect language without bothering about its relationships to other questions. Mathematics is no longer a monument but a heap."

The Joy of Lecturing—with a Critique of the Romantic Tradition in Education Writing

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1. A Sage on the Stage Speaks His Mind

A cornerstone of the current mathematics education reform is the recommendation that teachers should cease being "the sage on the stage", but should instead assume the role of "a guide on the side".[1] Lecturing is discouraged; direct instruction is passé. Students should be working in small cooperative groups to discover the mathematics for themselves, and the instructor should be merely providing guidance on the side. Indeed, "students frequently working together in small cooperative groups" is second among what an eminent educator considered to be the five preeminent characteristics of the present reform effort ([6, p. 105]).

It appears to me that this rejection of the sage-on-the-stage method of instruction is unjustified. There are situations where lectures are very effective, and in fact there are even circumstances which make this method of instruction mandatory. Furthermore, in recommending the guide-on-the-side strategy, educators should have been more forthcoming about its limitations so that teachers can better decide for themselves whether or not to follow such a recommendation. The purpose of this appendix is to amplify on these remarks. Although there are many alternative methods of instruction other than lectures, I shall limit the present discussion to the guide-on-the-side format on account of its favored status in the current reform.

It may be assumed that a person who rises to the defense of lectures must be someone who has never taught any other way; it would not be unnatural to go even further and conclude that the only way he can teach is by giving lectures. Not so in this case. Although I have been lecturing in the classroom for all thirty-three years of my teaching life, outside of the classroom I rarely give lectures in the sense of systematically presenting a body of knowledge. When undergraduate students come to my office with questions, for example, I do not believe a short lecture by me giving complete answers would do any good in an overwhelming majority of the cases. Instead I try to engage them in a dialogue and employ the Socratic method to expose for their own benefit the gap in their understanding that led to their questions.[2] In other one-on-one situations, I also do not lecture. I have given reading courses to undergraduates, and in such cases, I make clear that learning can only be achieved by the student and that all I can do is to nudge him in the right direction and offer help when absolutely

[1]Although this "sage-on-stage and guide-on-side" dictum has been in existence since the late eighties, it appears difficult to find a precise reference to it in the literature. One place where it is mentioned unambiguously is footnote 15 on p. 17 of [2].

[2]Unhappily, most students are only interested in getting simple answers and getting out of my office as fast as they can. My attempt at fostering genuine education only result in bad student evaluations for my "unfriendliness".

necessary. The student must do all the work and my contribution is essentially limited to asking key questions when we meet. It is the same when I find myself tutoring high school students on occasions. No lectures. The only thing I insist on is that no time limit be placed on any of the tutoring sessions: Once we start on a topic, we stay on course until it is finished regardless of how long it takes. For a later purpose, let me describe one specific example of my tutoring experience.

I once had to teach someone the division algorithm for polynomials. I started by remarking that it was just a glorified version of the same algorithm for integers. I asked if he knew the latter (yes), and if he could prove it (no). So I suggested a proof by induction of the division algorithm for given positive integers a and b in the form $a = qb + r$, $0 \leq r < b$. It took a while for him to decide, with some help from me, that one could attempt an induction on a, but in due course he succeeded in writing down a complete proof. Next came the polynomial version $f = qh + r$, $0 \leq \deg r < \deg h$. I asked him whether he could imitate the case of integers. It took some time for him to realize—again with some help from me—that $\deg f$ could be used for induction. However, he immediately saw that the usual induction step of "$P_{n-1} \Rightarrow P_n$" was of no use in this situation. At that point, it was time for me to step in to teach him about complete induction in the form of "$P_1, P_2, \cdots, P_{n-1} \Rightarrow P_n$". Then I let him figure out how to use it to prove the algorithm. Getting an appropriate q to start off the induction argument was not easy for him. While I saw the frustration, I left him alone because the frustration has to be part of the learning process. Finally he got it done. The whole session took something like two hours. I had no doubt that he really learned the algorithm through this tortuous process, and it is likely that for most students this is the only way to learn it. But could I teach in any fashion remotely resembling this in the usual junior level introductory algebra course? Absolutely not. In such a course, the polynomial algorithm merits a discussion of about 25 minutes. If I spent two hours to teach it, I would be fired for pedagogical turpitude,[3] and rightly so.

In an ideal world, I would like to teach all my classes the same way I teach my students in a one-on-one situation. But this is a dream largely unrealized except during the extra problem sessions I offer my students in upper-divisional courses. With a sparse attendance and little time pressure, I can afford to let the students dictate the pace and the direction of the discourse half of the time. Otherwise, I find the obstacle of time-constraint almost impossible to overcome, and this constraint will be a recurring theme of this article.

2. The Hows and Whys of Lecturing

No matter who says what, lecturing is *an* effective way of teaching in a university—and for that matter in grades 7-12—so long as our education system stays the way it is. Such a bald statement requires a careful description of its

[3] In my contract with the University of California, there is a clause that says I could be fired for moral turpitude.

underlying assumptions, and I will proceed to do that. I assume that:

> **(i)** The instructor is mathematically and pedagogically competent,
> **(ii)** Only 12 years are devoted to public school educations and 4 years to college education,
> **(iii)** After 12 years of school education students should be competent enough to function as useful citizens in society, and after 4 years of college students should be competent enough to start graduate work in their chosen disciplines, and
> **(iv)** Our education system continues to be one for the masses rather than for a select few, so that each teacher or professor must teach *many* students in each course.

Whatever I say below will apply equally well to teaching in grades 7–12, but for the purpose at hand I will specifically discuss only college teaching. The sage-on-stage style of lecturing has come under attack in the current mathematics education reform, but the attack seems to show no awareness of the basic constraints of **(i)**–**(iv)** above. For example, if the amount of material to be covered in a course can be greatly reduced (thereby violating **(iii)**) and students are expected to spend 8 years in college (thereby violating **(ii)**), then we can all safely abandon the lecture format and engage in a wholesale application of the guide-on-the-side philosophy in our teaching. To put this comment in context, let us continue with the above discussion of the polynomial division algorithm by considering it specifically as a topic in a junior level algebra course.

The purpose of a mathematics course is, naturally, to further students' knowledge of mathematics and logical reasoning skills, but there is also a practical aspect along the line of assumptions (ii) and **(iii)** above. Thus a junior level algebra course should enable its students to acquire a minimum mastery of the most basic techniques and ideas in algebra: the concepts of generality and abstraction, the concept of mathematical structure, and certainly the basic vocabulary of groups, rings and fields. The details may vary and the broad framework is susceptible to a certain amount of stretching (cf. [12]), but ultimately the course must serve to fulfill assumptions **(ii)**–**(iii)**. Students coming out of such a course should be ready to embark on more advanced courses in mathematics and the sciences, deal with the basic technical issues in industry, or at least be able to look back on the high school materials of polynomials, triangle congruence, or fractions with renewed understanding. Given that such a course would typically meet for only 45 hours (a semester), class time must be used wisely. This is the reason why only half a lecture can be allotted to the explanation of the polynomial division algorithm.

Learning mathematics is a long and arduous process, and no matter how one defines "learning", it is not possible to learn all the required material of any mathematics course in 45 hours of discussion. To make any kind of teaching possible, professors and students must enter into a contract. The contract can take many forms, but the following would certainly be valid: The professor gives an *outline* of what and how much students should learn, and students do the work on their own outside of the 45 hours of class meetings. Lecturing is one way to implement this contract. It is an efficient way for the professor to dictate the pace and convey his vision to the students, on the condition that students would do their share of groping and staggering toward the goal on their own. It should be clear that without this understanding, lectures would be of no value whatsoever to the students, especially to those who expect to come to class to be spoon-fed all the tricks for getting an A in the course. In advocating "guide-on-the-side" over "sage-on-the-stage", did educators weigh carefully the intrinsic merits of the lecturing format against the apathy of the students before putting the blame squarely on the former? Or is this simply a case of expediency over reason, because there are hidden forces at work which the educators did not bring to the table? Have they perhaps re-defined learning without telling us what they *really* have in mind? If so, then this is an illustration of what I call the *romantic tradition* in education writing: Unpleasant details are left to the imagination because they might interfere with the attractiveness of the advocacy in question.

Let us return once again to the polynomial division algorithm for a more detailed discussion. *Lecturing can take many forms.* In the 25 minutes or so allowed for the teaching of this topic in a junior level mathematics course, one way to approach it in the classroom is for the professor to indicate briefly the long process of possible trials and errors in arriving at the correct proof and to discuss the main points of the proof in precise terms. Most of the 25 minutes would therefore be spent on explaining the details of the induction on the degree. In order to understand such a lecture, students will have to retrace *on their own* the steps of the trials and errors omitted in class (see the discussion of the tutoring session in the preceding section). There are other ways to handle the lecture. For instance, if the textbook is reliable and readable, the professor may decide to let the students read the polished final account at home but use the class time to go through, as much as possible, the tedious learning process in the allotted 25 minutes. Or, the allotted 25 minutes of class time could be used to go through the initial segment of the trail and error process and, following which, students are told what they need to do in order to complete the investigation. For this kind of teaching to work, the students would have to be very mathematically mature. There can be other variations too. No matter. The fact remains that if we abandon lecturing and the underlying assumption of the sharing of labor between professor and students, and insist that the whole learning process (guided discovery, trials and errors, etc.) must take place within the 45 hours of class time, then the amount of material that can be covered in each course would be reduced by half if not more. Unless we stretch college education to 8 or 10 years, this is not a realistic option.

Last semester (Spring 1998) I taught a one-semester introductory analysis course, and I volunteered to give two additional problem-solving sessions each week. Attendance in these sessions was optional, and therefore poor.[4] Since

[4]I have volunteered these problem sessions often, and attendance has always been poor. Typically about 15-20%. I wish to let this fact be known.

in these sessions time pressure was not a serious concern, I could often indulge myself in my teaching method for private tutoring (see the preceding section). I did not insist on any kind of cooperative learning, but I let them decide for themselves if they wanted to discuss with their neighbors. Once I had about seven students, and I asked them to prove that the function $f(x) = \sqrt{x-5}$ is continuous at $x = 10$ by the use of ϵ and δ. Of course this requires a rationalization of the expression. The trick in question happened to have been discussed briefly several weeks before in the context of the limit of sequences, but it would appear that none of them had any recollection of it and, in any case, they could not make the connection.

I walked around the room, talking to each of them trying to coax at least one of them to come up with a reasonable plan of attack. After more than ten minutes of futility, it became clear that they had to be told. So I mentioned the word "rationalization", and one student immediately caught on. A few more explicit hints later, all of them knew what to do. The actually doing, needless to say, took a while at that stage of their mathematical development. I looked at each student's work and literally guided the hands of a few of them. After more than 15 minutes, they all got it done. Then I asked one of them to go to the board to give a complete exposition, and I followed with some general comments, partly to point out the pitfalls along the way and partly to bring closure. They probably learned something from the experience, but it must be pointed out that it took almost the whole 50 minutes of class meeting. At the risk of harping on the obvious, whatever might be the educational benefits of this way of teaching continuity, an introductory analysis course taught entirely—or just *frequently*—this way could hardly get off the ground.

There is another aspect to lecturing that deserves to be discussed. Lecturing allows the professor to share his insight with students beyond what is found in the textbooks. Again allow me to offer an example from my personal experience. Each time I teach calculus, I go through what I have come to call the "catechism of π". Most students believe they know what π is. To my question of "what is π?" usually comes the reply "circumference divided by diameter". So I ask "what is circumference?", the quick rejoinder of "2π times radius" is usually followed by nervous tittering. They know they have been had. Sometimes the catechism replaces "circumference" by "area", but the result is of course the same. Some years ago, I decided to address this issue directly by defining for them, right after the discussion of arclength, the number π as half the circumference of the unit circle:

$$\pi \equiv \frac{1}{2}\left(2\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx\right),$$

and then proceeded to *prove* for them that with this definition of π, the area of the unit disc is actually equal to π, *i.e.,*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = 2\int_{-1}^{1} \sqrt{1-x^2}dx.$$

This requires an integration by parts argument. Could I have guided my students on the side to the final conclusion? For 20% of them, maybe—that is just my guess— and only if I had two lectures at my disposal. But I had only half a lecture (25 minutes) to work with, because in the remaining 25 minutes, I also showed that the circumference of a circle of radius r is $2\pi r$, defined the *radian* measure of an angle correctly for the first time, and computed the arclength of a segment of a parabola together with an explanation of why Archimedes could find the area enclosed by the segment but not its arclength (he had no logarithm function, only polynomials). Such a pace is normal for a calculus course at Berkeley.

Now I do not wish to give the impression that each time my colleagues and I give a calculus lecture we always have this kind of interesting information to offer. How can this be true when so much spade work must be done in a basic course of this nature? Nevertheless, it is no stretch of the imagination to say that most of these lectures routinely carry elementary insights about mathematics which comes only with years of working at it. I see no reason why students cannot profit from such insight by making an effort to understand the lectures instead of adamantly resisting them by coming to class unprepared, or for that matter, leaving it without making an effort to understand it later. The teaching of calculus varies in each university, but to any conscientious lecturer, the preceding account must resonate to some extent. We hear that lectures are a relic of the past. Does this mean that it is improper for students to pick up essential information because they *have to* "discover" it by themselves? Or is it the case that, since students no longer want to put in the strenuous effort to learn, the universities must henceforth resolve to teach either the most simplistic aspects of mathematics or only the smallest possible amount consistent with the guiding-on-the-side philosophy?

For advanced (upper-division) courses in mathematics, the professor's understanding and vision of the subject are even more important in providing proper guidance to students—recall that we are assuming a large amount of material is supposed to be covered in each course (assumptions **(ii)** and **(iii)** at the beginning of this section). This is especially true in view of how textbooks are written these days (cf. [12] again). Lecturing is not the only way to provide this guidance but, until there is data to prove otherwise, it is one way of doing it. However, since the lecturing format is most heavily criticized in the context of the teaching of calculus, let me continue with the example of π and discuss calculus lectures. It has been said that the typical calculus lecture is *worthless, because it has virtually no conceptual development in it, is boring, and focuses mostly on techniques.* If this a judgment on the average calculus instructor's pedagogical or mathematical deficiency, then it has no bearing on our present discussion of lectures *per se.* On the other hand, learning about a correct definition of π and understanding for the first time how π enters into the circumference or area formula certainly gives a good account of the conceptual development in mathematics, makes interesting topics for students, and convincingly demonstrates how technique is inseparable from conceptual understanding in mathematics. Such being the case, it is clear that we have not even begun to exhaust the potential of lecturing. The sage on the stage still has work to do.

3. Time-Compressed Instruction

In the preceding sections, I have repeatedly emphasized the *time-compressed* nature of classroom mathematics instruction. In order to learn what is taught in class, students must be willing to spend two to three times the amount of time by themselves. For example, a survey conducted by the PDP (Professional Development Program) unit on the Berkeley campus in the 1980's shows that those students who got A's and B's in freshmen calculus spent an average of 10 to 14 hours on the course material per week outside of regular class meetings. To put these figures in context, since all the calculus courses are only 4 units each, conventional wisdom would have students spend only 8 hours per week instead of 10 to 14 hours. As another example, a recent article ([10]) makes the following comparison between the work habits of Japanese and American school students:

> Another after-school activity that occupies the time of adolescents is homework. Great emphasis is placed on homework as the basis of the excellent performance of Japanese adolescents in mathematics and science. In a typical survey, therefore, one might ask high school students how many hours they spend doing homework each day. The answer often given by Japanese students is unexpected: none. Only by pursuing the topic further does the actual state of affairs become clear. Additional discussion and questioning reveals that homework is often not assigned, but high school students are expected to spend several hours a night reviewing the day's lessons and anticipating the lessons for the following day.
> It is increasingly common in both middle schools and high schools in the U.S. that homework is done in school and simply represents work that teachers expect to be done before the next class meeting. The apparent lack of homework assignments was lamented by American parents and teachers. Parents questioned how their children could complete their homework during school hours, a practice very different from what they remember of their own school days. Teachers were concerned about the tendency of students to equate homework with studying; if there was no homework assignment, there was no studying.

Would it be fair to conclude from this that the bashing of lectures in the U.S. is a direct consequence of the infusion into our campuses students who are rarely asked to work outside of class all through K–12? In an inspiring address commemorating the centennial of the *American Mathematical Monthly* [11], Herbert Wilf bluntly stated: "In recent years, we have witnessed serious decline in the demands that we make on our students for intensive and solid intellectual achievement in mathematics. When we feed them more baby food every year, we thereby become accomplices to their intellectual softening." Wilf did not concern himself with the abolition of lectures, but he might as well have.

There are at least two special features about mathematics that dictate the time-compressed nature of mathematics instruction in the classroom: It is *cumulative* and it is *precise*. By cumulative, I mean that at any given point of a mathematical exposition, it is virtually impossible to understand what is taking place without first acquiring a thorough understanding of all that has gone on *before* that point. The failure to confront this rather brutal fact—in a mathematics class, once behind, forever behind—may be the single factor most responsible for the undoing of our mathematics students. The precision of mathematics stems from its abstract nature. Whereas even in a rigorous discipline such as physics, a photograph or a measurement by a laboratory equipment can render verbal explanations superfluous, the basic concepts of mathematics reside only in the realm of ideas and therefore must be meticulously described. Students must learn to concentrate fully on *every* word that is used in the description, or they run the risk of missing the point entirely. We all remember how as students we had to struggle with the seemingly innocuous quantifiers "for every" and "there exist" in the definitions of limit and continuity. And that is only for starters. The precision in mathematics *is* unforgiving indeed.

These two features make it difficult for students to learn from mathematics lectures if they are unwilling to also invest time and energy before or after the lecture for this purpose. In a videotape ([1]) made available by TIMSS (Third International Mathematics and Science Study) in 1997, two Japanese teachers were shown to give lessons—unequivocally based on direct instruction—with the rapt attention and active participation of their students.[5] Now that we have the preceding account ([10]) of the kind of preparation Japanese students routinely make before coming to class, we are finally in a position to understand why the teaching of mathematics in Japan achieves such good results and why their students always score so well in international tests. The last time I checked, the slogan of "guide-on-the-side but not sage-on-the-stage" has not been aggressively promoted in Japan.

4. The Importance of Being Honest

The folk wisdom that there is no free lunch in this world seems for some reason to be missing in current education writing, and this fact seems to be the genesis of the *romantic tradition* mentioned in §2. Wonderful new prescriptions for ailments of long standing in education are given on a regular basis with nary a hint of the likely detrimental side effects. On the K–12 level, for instance, "real-world" relevance of mathematics has been trumpeted as the salvation of the school mathematics curriculum *without* the caveat that unless this is done in moderation, the abstract nature of mathematics as well as its internal coherence would be jeopardized. Sure enough, the worst fears were realized in most (if

[5] In particular, no cooperative learning there.

not all) of the recent school mathematics texts, which emphasize "real-world" relevance (cf. p. 1535 of [8] and the references given therein).

The advocacy of the abolition of lectures as we know them is another case of promoting an idea without any explicit warnings of the possible losses and gains. For example, a more balanced approach to the subject of lecturing might begin by listing its strengths, its weaknesses, and the range within which it would be effective. A summary of the preceding discussion would include the following among the strengths of lecturing:

> **(a)** It allows the instructor to set the pace of the course. This is an important consideration if the basic parameters of school and college education as we know them are to remain intact. See assumptions **(ii)** and **(iii)** of §2.
> **(b)** It allows the instructor to share his insight into the subject with students. If we still believe that education is the process of passing the torch from generation to generation, this too is an important consideration.

A side benefit of the lecturing format, one that has not been discussed thus far, is that it forces students to stretch their concentration spans. In these days of MTV when everything is interactive and instantaneous, such a beneficial effect should not be dismissed lightly. In fact, one can speculate on the possible correlation between the onset of the computer-age and the dissatisfaction with lectures. As to the weaknesses of lecturing, the most serious is that, unless students are willing to do their share of the work outside the class and meet the instructor halfway, lectures are a waste of time. It is possible that this aspect of the lecturing format has never been made explicit to a large percentage of our college students, and the dismal student performance in mathematics courses is the result. The guide-on-the-side advocacy then becomes a facile, one-dimensional response to a multifaceted challenge. One reason why lecturing has been the accepted mode of instruction in most universities for so long is probably the assumption that students are there to work. Are we at the dawn of a new era when even such standard assumptions must be re-examined? Perhaps universities can no longer survive as institutions of higher learning but must transform themselves into "caring, nurturing", glorified high schools.

We now come to the guide-on-the-side method of instruction which, as mentioned, means the guided discovery method *in the context of cooperative learning*. What are its implicit assumptions and what are its strengths and weaknesses? By transferring what used to be activities outside of class into the classroom, the discovery-via-cooperative-learning pedagogy tacitly assumes either that students can no longer be trusted to do their share of the work or that they are incapable of doing it. The great advantage of this method of instruction lies in its seeming ability to make mathematics accessible to a much wider audience than is possible in the lecturing format.[6] The slower

[6]But by no means to ALL students. I will not reproduce here the oft-repeated anecdotes about how some members of a study group sit and do nothing while one or two members take charge and do all the discoveries for them.

students who do not wish to put much energy into a mathematics class would certainly find participating in cooperative learning more congenial than listening to lectures. On the debit side, guided discovery and cooperative learning slow down the pace of a course, at least by half. One may surmise that the authors of some textbooks which advocate this particular pedagogy are well aware of the attendant loss of class time, and therefore deliberately set out to cut down on the more mathematically substantive topics. Thus we find calculus texts which do not even present the proof of something as basic as the Fundamental Theorem of Calculus (cf. [7] and [9]). Another drawback of this particular pedagogy has also been discussed: A guide-on-the-side has fewer occasions to share his vision or insights with the students than a lecturer. Those who would otherwise profit from the knowledge and experience of their instructors end up being short-changed by this pedagogy. If we look past the heroic efforts of a few extremely talented instructors, it would be fair to say that students in currently advocated programs of guiding-on-the-side typically learn the details in a small area but not acquire much of a perspective overall.

In this context, an additional comment about the possible omission of topics in a guide-on-the-side classroom may not be out of place. It is a fact that American high school graduates are among the least mathematically knowledgeable compared with their counterparts in nations that did well in TIMSS (cf. [2]–[4]). We can also verify directly from our own collective experience that American students are generically the least prepared among our graduate students. Would it not be fair to say that our undergraduate mathematics curriculum is already down to the bone and has no more fat to be trimmed?

The preceding discussion of lecturing and its common alternative is by no means exhaustive, but even this much critical analysis would have been beneficial to the current debate on the mathematics education reform. For instance, where in this advocacy for guided discovery in the classroom do we find an explicit reference to the underlying assumption about the students' unwillingness or inability to work on their own? (Consult [11] again.) Or is it the case that this assumption is a misapprehension? These issues should have been openly debated long ago so that teachers who opt for one or the other pedagogy would have the benefit of knowing what they are getting into. It would be wrong to say that this advocacy has produced nothing of value thus far. Quite the contrary. Because of this advocacy, some of us who had to struggle to become mathematicians—and have always assumed that all students must know the need to do the trials and errors on their own—have been awakened to the fact that we must *tell* the students of this need or even *demonstrate* to them this need by use of examples. But given the human tendency to oversimplify, the danger of a passionate advocacy in a subject such as pedagogy—which is far from a hard science as of 1998—is that blind acceptance and a reckless pursuit would inevitably follow. The classic dictum that if a little bit is good then a lot must be better unfortunately applies only too well in this situation.

Lest this article sound like a defense of the status quo of the lecture format, let it be said—however briefly—that perhaps the quality of some lectures does raise legitimate concerns. There are lecturers who fail to observe the basic etiquette of lecturing (cf. §§1.6 and 2.13 of [8]), and there are also those who still cling to textbook-writing-on-the-boards as a legitimate form of lecturing in spite of the present super-abundance of adequate textbooks on almost every standard topic. For lecturing to survive, the practitioners of this art must continue to be vigilant (see assumption **(i)** of §2). Nevertheless, the over-riding fact remains that the current discussion of pedagogy fails to meet the most basic requirements of scholarship: Any advocacy should state clearly its goal, its benefits, and its disadvantages. In this light, the advocacy of the guide-on-the-side pedagogy has been presented more like an info-mercial than a scholarly recommendation. It is all good and nothing bad could possibly come of it.

In the field of medicine, the FDA has made the listing of the precise range of applicability and the side effects of each drug mandatory. Would it be too much to ask that the same consideration of fairness be also extended to teachers so that all education writings are always accompanied by an analysis of the limitations of a particular proposal, including its drawbacks and the conditions under which it would not be applicable?

Acknowledgement. I wish to thank Ralph Raimi, Dick Stanley, and Andre Toom most warmly for their invaluable criticisms of an early draft of this article. In addition, Roger Howe's pithy observations led to significant improvements.

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Teaching Freshmen to Learn Mathematics

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From a mathematics instructor at another university:

"I have just finished reading your article [T.U.L.] in the Notices of the AMS.[1] I absolutely agree [with] every word. I taught some undergraduate courses [here] getting extremely good student evaluations. First I was very surprised, then the next semester the ratings were even higher ... and I still didn't understand.

"Now in the light of your article it makes sense: I taught like this were a high school."

From *The sum of mediocrity*, an article on pre-college education by Pat Wingert, in the December 2, 1996 issue of *Newsweek*:

"We expect less from our students, and they meet our expectations."

One thing that should be happening during the first semester of the freshman year is that the students start to figure out how to learn on their own, and get beyond skimming the surface of the subject. This is the main academic adjustment that most students must make when they get to college. We should therefore insist that they do it. This could involve running our calculus courses in a way that is very different from what they are used to from high school. A simple illustration of this is insisting that the students read the textbook, both for concept and examples; we will get back to this later.

At the beginning of the Spring '97 semester, I was talking to a student from my Fall '96 Calculus II (Physical Science and Engineering) course. I had gotten to know her through visits to my office hours for help in getting started on some of the assigned problems. For instance, at the beginning of the course she was unable to make sense of the problem: Show that when n is a multiple of 4, the sum $1 + 2 + 3 + \ldots + n$ is an even number. Later, she could not figure out how to deal with a sequence defined by a simple two-step recursion. After failing Exam 1, she told me about being panicked during the exam, and she sought my advice. I told her to engage a more secure comprehension of the material, so that it wouldn't get lost under the strain of examination conditions. Her performance on subsequent exams was satisfactory, and her course grade was a C. (I would describe the exams as pretty straightforward, but very thorough.) This is just one example of evidence that the course is being run at a reasonable level for our students.

I felt that my course had helped her substantially, and I sought confirmation from her. What she told me at first I'd heard before: she didn't like what I was doing in the beginning. *After the semester was over*, she said, she had looked back and realized that the way I conducted the course forced her to learn how to

[1] The article was titled *Teaching at the University Level.* It appeared in the August 1996 issue of the Notices. Letters to the editor about it were published in the November 1996 issue; a letter of mine appeared in the December 1996 issue. We reprint as an appendix to the present article the appendix from T.U.L., *Academic Orientation* [A.O.], with mild editing. Item #7 from A.O., omitted in the August issue, has been restored and enhanced.

learn. Moreover, I was astonished to hear, mine was the only one of her courses that did![2]

In a way, she was a good student. Before continuing with this, I want to describe the backdrop. The students in the course were almost all freshmen with A.P. credit for Calculus I. The review of the most pertinent material from Calculus I was left to the students in the first homework assignment. The handling of integer variables, material that would become relevant later for sequences and series, was treated. An important thing presented in the first week of lectures was the mathematical usage of "if/then" and "only if"—a majority of the students seem to think at first that "if" means "if and only if"—so they could read the textbook correctly. A good way to confront the issue is by showing the two sentences from conversational English:

a) *If it stops raining, I'll go to the store.*

b) *If I win the lottery, I'll buy a new car.*

These have parallel structure but different connotation. It is not hard to convince the students that we cannot afford such ambiguity in mathematical writing.

The assignment for the next week consisted of three problems designed for "consciousness-raising" (in a course where the students had been told to expect 8 hours of work per week outside of class). One of these was the problem about $1 + 2 + 3 + \ldots + n$ mentioned above. A better one, I think, was

True or false, and explain fully (i.e., verify or give a counterexample):

a) $f(x)$ *is a rational function only if* $\int f(x)dx$ *is rational.*

b) *If* $f(x)$ *is a rational function,* $f'(x)$ *is a rational function.*

The answers to these are familiar from Calculus I experience, but few of the students had ever thought about the processes of differentiation and integration in the large. To guide them, I went through the verification that the sum of two rational functions is a rational function. Also, I explained in lecture what is meant by a counterexample. They were instructed to write up their solutions carefully. Most of the students discover that they can do such problems if they persevere. While it seems hard to squeeze them in, problems of similar depth should be given as the course progresses.

To return to the student in my office, I asked her to evaluate $1+2+3+\ldots+n$ in order, for $n = 1, 2, 3, 4$, and emphasized the parity of the answers. That already triggered something in her head, and she was prepared to persist with

[2] A likely explanation for this is given, more or less, in T.U.L. Untenured faculty are afraid of incurring complaints about their teaching that could hurt their careers. Tenured faculty have lost their will to stand up to student indifference and resistance; besides, doing so can also cost in salary raises, given the way teaching is often evaluated.

the problem. A bit later, she reported something very "unusual": As she spent more and more time with it, she saw her understanding start to grow. I assured her that this was very normal. The expectation that the answer is either there or not there is one of the many misconceptions that freshmen have about mathematics.

I have been teaching that Calculus II course every fall semester since 1993. In '94, I changed my attitude toward my role as instructor,[3] substantially increasing the amount of thought during the preparation and energy into the delivery of the lectures, and additional effort in printing up handouts to supplement the lectures as needed. But I also expected the students to match it with increased learning; the aspiration had become command of the material. Though the results were good, there was still a lot of grumbling in the class. Indeed, the libelous review of that course in the student course guide led to the shutting down of that publication. My aspirations have not changed much since 1994, though my understanding of the educational issues has.

One of the students in the 1994 course, who also reported being unhappy at first, wrote of the lectures, "He made the material very easy to understand, if and only if you were doing the work necessary to keep up with the class." I think he was trying to convey the message that a student who was not keeping up would be unable to see that the material was being explained in a clear and helpful manner. It reflects poorly on the current state of affairs that I started to wonder whether that was fair! With the support of my department chair, I decided that it was. I was encouraged further by the dean of the college, now provost (whose academic credentials are in the humanities, by the way).

I remember vividly when, during the second or third week of the 1993 course, a student came to see me, feeling that he was hopelessly lost. I probed with a few questions, after which both of us could see that he was very close to understanding the material. He, like so many other high school graduates, had been trained to absorb mathematics in tiny controlled doses, which are to be memorized and later regurgitated. It is no wonder that the suggestion of learning concept often gets perceived by students as irrelevant theoretical digression, rather than the means to better comprehension that it is.

It struck me just a couple of weeks before Fall '95 began that there was no clear way for the students to figure out what it was, so different from their high school experience, that I wanted them to do. Some students make the transition instinctively,[4] while others simply blame the instructor for teaching poorly. This led me to do some serious orientation for my own students that year. It was a nice coincidence that during the first week of classes, I crossed paths with a senior who had taken Calculus I from me in his freshman year. I started telling him about my ideas on academic orientation. At one point he said to me "You have to remember that they are freshmen, and that they don't

[3] I should thank my colleague W. Stephen Wilson, who was Chair at the time, for pushing me in this direction.

[4] A strong student reported that it took her a couple of weeks to get into stride.

realize that whatever they think of their instructor, they'll be learning most of the material on their own anyway." I recall that I came back with *"Why don't they know that?!"* If the high schools and older students aren't communicating that, then it has to come from us.

I then pressed hard for academic orientation in the university. For obvious reasons, it is better that the message have the ostensible endorsement of the university: of the math department and the academic deans. I ended up giving a presentation on mathematics during Orientation Week, 1996, a first. At Hopkins, it comes down to communicating to the entering students what most sophomores, and virtually all juniors and seniors, know about education in college. This involves making the new rules explicit (see the appendix Or-2 *Academic Orientation* [A.O.]), and debunking the misconceptions that so many of the freshmen bring to college. Appended as Or-1 is a compilation of such misconceptions. The first five items were discussed first to help soften the impact of the potentially startling A.O., which was circulated about half-way through the presentation.

I am convinced that the level at which my course is now conducted is about right (for its audience). If we assume that this is correct, it is hard to justify on educational grounds running the course at a lower level. My university has 900 to 1000 entering students each year, who are ambitious (at least in the abstract) and "bright" (mean SAT Math score a little over 700). Most of them did not have to work hard in high school; here lies a big part of the problem. One of my 1996 students, who *was* used to having to work hard, reported what another student had said: "I'm so pissed off! I got a $C-$. I couldn't believe it. I never worked in high school, and I always got A's!" And that was *after* orientation! Nobody said that our task is an easy one.

Because some students can't, most—around 85%—of our students were never asked to learn mathematics in high school by reading from a textbook. They grew accustomed to picking up the material from classroom presentation alone, even in A.P. courses. But in fact, the students who attend a good college are capable, with some exertion, of reading the book; we therefore want them to do it. For some, it's a struggle. Even the stronger students will encounter things in the book that they can't, or just don't, figure out. However, there are numerous ways they can have this straightened out, viz., lecture, section meeting, TA office hours, help room, professor's office hours, discussion with classmates, ... Given the advent of extended help room hours, I find even more outrageous the suggestion that the lectures be aimed at those who want to skimp on their effort in learning the material.

When college students say they want a good teacher, some want a good educator, one who will help them to make it *through* the material. They accept that inherent to mathematics is the need to decide what to do to solve a problem, to make distinctions and choices, to reject ideas that are fallacious, and to persist when one's first attempt doesn't work. This flexibility[5] is often absent from high

[5] Intellectual flexibility is a notion that makes sense in *all* disciplines. As such, it is wise to talk about it when we explain to educators in non-scientific fields what we want from our

school experience, where mathematics is taught largely by repetition. As such, they were trained to learn the subject *inflexibly*. Many students want it to stay that way in college; they want to be helped *around* the material, in effect, bailed out by the instructor from having to understand it. Pandering to the latter group is slowly but surely eroding the quality of mathematics education in American colleges and universities, and even abroad.

An interesting thing I learned in 1996, when the course was divided into two lectures, with the other run in tandem with mine by a remarkably skillful assistant professor, was that the freshmen here will praise an instructor who runs the course at a high level provided the lectures are well-organized, focused, and "self-contained". However, it does not follow that they learn better; his students and mine performed comparably on exams (cf. misconception #13). All too many students want to come into class cold, expecting to get "taught" by the professor, from the ground up.[6]

What is the point of the instructor's commitment of time and effort to attempt to supply teaching that will be judged to be "better" when it is not getting matched by demonstrated better learning? I would go further. *No style of teaching mathematics can substitute for insisting that the students pick up their share of the work*, unless one is willing to compromise standards. We should stop seeking panaceas that place the burden on the competent instructor; I do not believe in the "Fountain of Education", and it is time to stop looking for it!

How can we fulfill our role as educator? The main theme is that one must aim for the determination of the appropriate level for the course, one that matches the level of the students who take the course *in one's own university*. Of course, I do not mean here the level that we see when the students follow their high school instincts, thinking that they won't be reading the textbook, or equating learning to memorizing a list of formulas, or declaring that spending three or four hours on homework is a lot of work. We must then have the conviction, and ideally the backing of the department and the administration, to hold to that level. However, in doing this we also put a greater obligation upon ourselves, for the threshold requirements[7] for the instructor, in giving a course at a higher level, are correspondingly higher.

Academic orientation is necessary, of course. The students must be told how things are going to be different from high school. This is more likely to succeed if there is a web of support that will block the students from resisting "because their professor has these crazy ideas." As reasonable as we may find the statements of *Academic Orientation*, most freshmen are shocked by them; they even wonder if it's for real. It doesn't help matters if some of our colleagues teach calculus as though it were "grade 13 of high school", an easy way to endear

students in mathematics courses.

[6]That would serve to keep the level of the course down, unless the students are expected to pick up a lot more as they read the textbook later. Students here are capable of learning the easier things in the course largely by themselves, and I remind them of that.

[7]This notion is mentioned in misconception #13. I know no algorithm for determining where the threshold is for a given level of aspirations and a given student body.

oneself with freshmen.

On the other hand, many students *have* heard in high school such things as "Don't just memorize. Learn concept." However, they found that they could score well on tests by ignoring this advice and behaving as they were advised not to. That must stop in college. Another big problem is that many freshmen wrongly believe that adjustment is needed only for students other than (weaker than) themselves.

To put the preceding into effect, we must be free to give the students what they need, not what they say they want. But if we do so, we risk lowering our ratings in the course evaluations,[8] for students who retain their belief that they are entitled to do well without exerting themselves are not going to be happy. This places us in the ironic situation that we might be penalized just for doing our job conscientiously.[9]

I'll summarize what I have been doing toward upgrading the freshmen's expectations.[10] When I gave a forceful presentation in my course in 1995, the Department had circulated J. Martino's *Survival Guide* to all students taking a large lecture course, and that reinforced my message nicely. In Fall 1996, my presentation was on the Orientation Week program, and that "implied" backing by the University. In Fall 1997, my previous experience enabled me to carry out efficient major orientation for my own course. I appealed to something I had ascertained was mentioned by deans of both the College of Arts & Sciences and the Engineering School in their addresses to the freshmen: the amount of work outside of class expected in a college-level course (cf. *Academic Orientation*, #3). After reminding the students of that during the first lecture, I told them, "I don't want to hear any griping about the workload in this course unless you are consistently putting in more than 8 hours a week. And if you are not highly talented, you may decide that you *want* to put in more." I feel that I've told the students this year everything I might want to say in the way of orientation.

Above all, the students should get the message that we intend to help them learn the material, but we are not going to bail them out if they don't. The exams in the course must uphold the level announced for the course. They must be made up so that they (de facto) penalize students who insist on operating on the basis of high school notions that we have declared inappropriate. The problems should be taken from the heart of the material, not from the surface (as students learn to expect in high school). No practice exams to suggest programming. I tell them that doing new problems of sufficient difficulty with books closed comes closest to simulating the exam situation. In lecture this year, I asserted before the first exam, "If you take all of the homework problems, examples done in lecture and the book, problems from past years' exams

[8] Actually, my own ratings went up from 1993 to 1994.

[9] I know that some (many?) past students (including, of course, the one mentioned at the beginning of the article) came to realize how much they benefited from my course only *after* the course was over, e.g., while they were taking Calculus III. "The world rewards the appearance of merit oftener than merit itself." –F. de La Rochefoucauld

[10] What one can do in the classroom is based in part on one's nature; it is unreasonable for ourselves or others to regarded us as teaching machines. Also, it may be relevant here that I had control of the entire course.

[available to the students], the problems on the exam will be different from all of them; but they can be done by the same methods."

The messages of academic orientation must be reinforced throughout the semester, and frequently in the first part of the course. Here's a sample of what my class heard or read this year:

— Talent and background will make some difference, but *you are going to have to work in order to succeed*, both in this course and in pursuing your career goals. If you choose to shoot for less, you do so at your own risk.

— The goal is to reduce your dependence on the instructor.

— Think about it after class. You should know by now that you don't *have* to understand it here.

— It's *impossible* for me to explain that to you. Some things you must try to figure out for yourself.

— Mathematics is about concept, attitude and control.

— The purpose of the exam questions is to determine the extent of your command of the material. Though you should be getting the correct answer to the problems if you have good command, it is not the main point. After all, if I only wanted to see the right answers, I'd just do the problems myself!

A strategic point: We shouldn't overlook the power of negative reinforcement. In Calculus II, the following lines of "reasoning" are all too common:

a) Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges. *Well,* $\frac{1}{n} \to 0$, *so the series converges.*

b) Compute $\lim_{n\to\infty}(1+\frac{1}{n})^n$. *Well,* $1+\frac{1}{n} \to 1$, *and* 1 *to any power is* 1. *The limit is* 1.

These are not just silly mistakes; they are fundamental errors that show disrespect for the methods of the course and for the instructor. (Why didn't we teach them the *"easy"* way to do it?) Indeed, it is essential to make the student see that these are wrong.[11] I term **a)** "the ultimate sin", and **(b)** the penultimate; it is announced that committing the penultimate sin in a problem gets an automatic zero credit, and the ultimate sin gets *negative* credit. These sins occur with surprisingly low frequency in my course; the class performs better now with infinite series.

[11] Students sometimes object to the idea of showing them that a way of "thinking" is wrong (rather than programming them with the right way to do the problems). I'm quite sure that every math instructor has the experience that students, having been shown the right way, make such mistakes anyway.

I should mention that the immediate reason for my writing "Teaching at the University Level" was that I felt the calculus reform movement was gaining too much momentum. The notion that students at universities like Harvard and Stanford needed reformed calculus was out of line with my own observations. Moreover, it is unlikely that today's college students are less intelligent than their predecessors; rather, something has happened to their sense of learning. While the transition from high school to college has always been a hurdle for students, today's students find themselves in a weaker position to deal with it. Since I also refuse to believe that they have been irreparably damaged, trying to repair the damage makes far more sense than pretending that the subject needs "genetic engineering". In particular, we don't have to wait for the difficult underlying problem with pre-college mathematics education to get resolved before we can start to remedy the situation in the colleges.

It is not my intention to condemn the calculus reformers outright. But if we are to get serious about resolving the difficulties our students are having in learning mathematics, we must first address the issues that really matter most, namely the low expectations that many of them hold when they arrive in college and their overestimation of the effort that they are putting in. Only then does it make sense to judge the merits of different methods of instruction . . . for a given group of students.

Appendices: Orientation material

Or-1. Misconceptions (and Rejoinders)

A. Common Misconceptions about High School and College

1. *Only a jerk could get less than a B-grade in a course!*

If that's true, then nearly half of the freshman in math courses in previous years are jerks!

2. *I had a good math teacher in high school who taught at my level.*

But most of the freshmen admitted here say they were in the top quarter (say) of their math class, yet agree that a teacher [in high school] is supposed to make sure that even the weakest students in the class learn. Now, whose level was the course run at?

3. *I did well in math, even calculus, in a good high school. I'll have no trouble with math at Hopkins.*

There is a different standard at the college level. The student will have to put in more effort in order to get a good grade (or equivalently, to learn the material sufficiently well by *college* standards).

4. *My AP calculus course in high school was like a calculus course in college.*

The student here is expected to do much more learning outside of the classroom; see also #3 above. (On the other hand, the Advanced Placement Tests are college-level exams.)

5. *College will be like high school, just a little harder.*

(See all of the above)

B. Misconceptions about Learning Mathematics

6. *In a calculus course, theory is irrelevant, for what is really at stake is doing the problems. The lectures should be aimed just at showing you how to do the problems.*

We want you to be able to do all problems—not just particular kinds of problems—to which the methods of the course apply. For that level of command, the student must attain some conceptual understanding and develop judgment. Thus, a certain amount of theory is very relevant, indeed essential. A student who has been trained only to do certain kinds of problems has acquired very limited expertise.

7. *The purpose of the classes and assignments is to prepare the student for the exams.*

The real purpose of the classes and homework is to guide you in achieving the aspiration of the course: *command of the material.* If you have command of the material, you should do well on the exams. On the other hand, some students act as though the exam problems have been decided in advance, and expect the lectures and assignments to be leading up to performance on *those* problems, or ones just like them. The latter would constitute the avoidance of our goal.

8. *The best way to study mathematics is to just memorize everything very carefully.*

As a colleague in the Physics Department once put it, "You can't memorize problem-solving!" Here, problem-solving refers to the ability to take a problem and attempt to carry out whatever methods might be relevant to solve it. This is a skill that grows with experience. (You might keep in mind as an analogy that memorizing the dictionary of a foreign language is not enough to achieve fluency in that language.)

9. *Students learn best when everything they have to know is presented slowly in the classroom.*

If everything the student has to know is presented slowly in the classroom, the total amount of material in the course will be rather little. Thus, students actually learn *least* that way.

10. *It is the teacher's job to cover the material.*

As covering the material is the role of the *textbook*, and the textbook is to be read by the student, the instructor should be doing something else, something that helps the student *grasp* the material. The instructor's role is to guide the students in their learning: to reinforce the essential conceptual points of the subject, and to show the relation between them and the solving of problems (cf. #6).

11. *Since you are supposed to be learning from the book, there's no need to go to the lectures.*

The lectures, the reading, and the homework should combine to produce true comprehension of the material. For most students, reading a math text will not be easy. The lectures should serve to orient the student in learning the material.

12. *A good teacher is one who can eliminate most of the struggle for the student, making the material easy to learn.*

Of course, it is possible to direct the students toward correct ways of thinking, but a certain amount of struggle is inevitable. Experience cannot be taught. Moreover, many topics are inherently difficult so they cannot be understood either passively or quickly. Eliminating the struggle can only be achieved by excising substance from the course (e.g., constricting the scope of the course, or reducing the means for recognizing where the methods of the course apply). Then the fraction of the material that remains could well be easier to learn, but the student will be acquiring diluted skills.

13. *When the students are happy with the instructor's lectures, they learn the material better.*

This statement is wishful thinking. According to the evidence I've seen, once threshold requirements are met the perceived quality of the instructor makes little, if any, difference in learning. What makes a real difference in learning is appropriate effort *by the student.* The best thing that a decent teacher can do, in order to get the students to learn better, is to hold high yet reasonable expectations of them.

Or-2. Academic Orientation

What follows is what an entering freshman should hear about the academic side of university life [in mathematics (and the sciences)]. It is distilled from what I've learned and written concerning the need for academic orientation.

The underlying premise, whose truth is very easy to demonstrate, is that most students who are admitted to a university like JHU were being taught in high school well below their level. The intent here is to reduce the time it takes for the student to appreciate this and to help him or her adjust to the demands of working up to level.

1. **You are no longer in high school.** The great majority of you, not having done so already, will have to discard high school notions of teaching and learning, and replace them by university-level notions. This may be difficult, but it must happen sooner or later, so sooner is better. Our goal is for more than just getting you to reproduce what was told to you in the classroom.

2. Expect to have material covered at *two to three* times the pace of high school. Above that, we aim for greater command of the material, esp. the ability to apply what you have learned to new situations (when relevant).

3. Lecture time is at a premium, so must be used efficiently. You cannot be "taught" everything in the classroom. **It is *your* responsibility to learn the material.** Most of this learning must take place *outside* the classroom. You should willingly put in two hours outside the classroom for each hour of class.

4. The instructor's job is primarily to provide a framework, with *some* of the particulars, to guide you in doing your learning of the concepts and methods that comprise the material of the course. It is not to "program" you with isolated facts and problem types, nor to monitor your progress.

5. You are expected to read the textbook for comprehension. It gives the detailed account of the material of the course. It also contains many examples of problems worked out, and these should be used to supplement those you see in the lecture. The textbook is not a novel, so the reading must often be slow-going and careful. However, there is the clear advantage that you can read it at your own pace. Use pencil and paper to work through the material, and to fill in omitted steps.

6. As for *when* you engage the textbook, you have the following dichotomy:

a) *[recommended for most students]* Read, for the first time, the appropriate section(s) of the book *before* the material is presented in lecture. That is, come prepared for class. Then, the faster-paced college-style lecture will make more sense.

b) If you haven't looked at the book beforehand, try to pick up what you can from the lecture. Though the lecture may seem hard to follow (cf. #2), absorb the general idea and/or take thorough notes, hoping to sort it out later, while studying from the book outside of class.

7. It is the student's responsibility to communicate clearly in writing up solutions of the questions and problems in homework and exams. The rules of language still apply in mathematics, and apply even when symbols are used in formulas, equations, etc. Exams will consist largely of fresh problems that fall within the material that is being tested.

Or-3. Two Telling Tales

1. Analogy: French in high school and college. I knew that the first-year French course (French Elements) covers about the same material as the first two years of high school French. This is typical of first year college language courses. Also, the semesters are shorter here, and one can calculate that the material is covered approximately three times as rapidly here as in high school.

After looking at the catalog description of French Elements, I called the instructor. I felt sure that there was more to it than just the triple speed. "Yes," she replied. "In our course, we aspire for fluency."

I admit that I had four years of French in high school, but no one ever spoke of *fluency*. It should be obvious that most of the work must occur outside of class. You can expect something like the tripling of high school pace, a lot of work outside of class, *plus aiming for the mathematical analogue of fluency* (perhaps *command* is the correct word), in a calculus course here.

2. Analogy: Martial arts. An 18-year-old enters a *tae kwon do* studio, walks up to the instructor, and states proudly, "I want to learn how to put my hand through a stack of bricks!"

The instructor thinks a moment, then replies. "Well, that's very difficult, and will take time. First, you must develop self-control and mental discipline. Then ..."

The youth interrupts, "Don't give me that discipline crap! Just teach me how to put my hand through the bricks!"

The instructor walks away shaking his head, as does the would-be student. One of the regulars of the studio, who teaches math in a local high school, steps up to the instructor. "You know, the young man has a point. All you have to do is make the bricks out of softer material, and crack them a little in advance."