**Symplectic groupoids of log symplectic manifolds**
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**Theorem 1** ([1]) Let $\mathcal{H} \cong D$ be a closed Lie subgroupoid of $\mathcal{G} \cong M$ over the closed hypersurface $D$, and define

$$[\mathcal{G}:\mathcal{H}] = \mathbb{B}_{\mathcal{H}}(\mathcal{G}) \setminus (s^{-1}(D) \cup t^{-1}(D)), \quad (1)$$

where $s^{-1}(D)$ (resp., $t^{-1}(D)$) is the proper transform of $s^{-1}(D)$ (resp., $t^{-1}(D)$). There is a unique Lie groupoid structure $[\mathcal{G}:\mathcal{H}] \cong M$ such that the blow-down map restricts to a base-preserving Lie groupoid morphism $p : [\mathcal{G}:\mathcal{H}] \to \mathcal{G}$. The Lie algebroid $\text{Lie}([\mathcal{G}:\mathcal{H}])$ is the elementary modification of $\text{Lie}(\mathcal{G})$ along $\text{Lie}(\mathcal{H})$, i.e. $\text{Lie}([\mathcal{G}:\mathcal{H}])$ has sheaf of sections defined by

\[
\text{Lie}([\mathcal{G}:\mathcal{H}]) = \{ X \in \text{Lie}([\mathcal{G}]) \mid X|_{\mathcal{H}} \in \text{Lie}(\mathcal{H}) \}.
\]

**Birational construction**

**Definition 2** A log symplectic manifold is a 2n-manifold $M$ with a Poisson structure $\pi$ whose Pfaffian, $\pi^\wedge$, vanishes transversely. Moreover, $(M, \pi)$ is proper if each connected component $D_i$ of the degeneracy locus $D$ is compact and contains a compact symplectic leaf.

**Theorem 3** ([2]) For a proper log symplectic manifold, each $D_i$ is a symplectic mapping torus. In particular, $f_2 : D_i \to \gamma_i$ is a symplectic fibre bundle.

**Gluing construction**

The restriction of a Lie groupoid $\mathcal{G} \cong M$ to an open set $U \subset M$, denoted by $(\mathcal{G}|_U)$, is the source-connected part of $s^{-1}(U) \cap t^{-1}(U)$. An orbit cover of $\mathcal{G} \cong M$ is a locally finite cover $\{U_i\}_{i \in I}$ of $M$ such that each orbit of $\mathcal{G} \cong M$ is contained in $U_i$ for some $i \in I$. If $\mathcal{G} \cong M$ is source-connected, then $\{U_i\}_{i \in I}$ is also an orbit cover for the underlying Lie algebraic $\text{Lie}(\mathcal{G})$.

**Theorem 4** ([1]) For a proper log symplectic manifold $(M, \pi)$, the symplectic pair groupoid $\text{Pairs}(M)$ is the adjoint symplectic groupoid, and $\text{Pairs}(M)$ is the source-connected part of $s^{-1}(D) \cap t^{-1}(D)$.

**Theorem 5** For an integrable Lie algebroid $A$ with an orbit cover $\{U_i\}_{i \in I}$, let $\mathcal{G}_i \cong U_i$ be a source-connected Lie groupoid and let $\phi_{ij} : (\mathcal{G}_i|_{U_i})^\gamma \to (\mathcal{G}_j|_{U_j})^\gamma$ be groupoid morphisms satisfying $\text{Lie}(\phi_{ij}) = \text{id}$, $\phi_{ii} = \text{id}$, $\phi_{ij} = \phi_{ji}^{-1}$ and the cocycle condition $\text{Lie}(\phi_{ij}) = \text{id}$. Moreover, every source-connected groupoid is obtained in this way.

**Example: log symplectic surface**

For example, we associate a graph (below) with the log symplectic surface (above).

In addition, we label the vertices and (half-)edges with the fundamental groups of $V_i$, $D_j$ and the symplectic leaf of $D_j$, and the kernel of the first Stiefel-Whitney class of $ND_j$ with the induced morphisms, as illustrated below.

The symplectic groupoids are classified by a family of normal subgroups for each of $\mathbb{Z}$, $\langle a, b \rangle$ and $\mathbb{Z}$.

In higher dimensions, Theorem 6 implies the source-simply-connected groupoid is Hausdorff if and only if, for each symplectic leaf $F$ contained in $D$, and for each class $\gamma \in \pi_1(F)$ on which the first Stiefel-Whitney class of $ND$ vanishes, the push-off of $\gamma$ is nonzero in the fundamental group of the adjacent open symplectic leaf or pair of leaves.

**Reference**
