Constructions of Lie Groupoids

by

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Abstract

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In this thesis, we develop two methods for constructing Lie groupoids.

The first method is a blow-up construction, corresponding to the elementary modification of a Lie algebroid along a subalgebroid over some closed hypersurface. This construction may be specialized to the Poisson groupoids and Lie bialgebroids. We then apply this method to three cases. The first is the adjoint Lie groupoid integrating the Lie algebroid of vector fields tangent to a collection of normal crossing hypersurfaces. The second is the adjoint symplectic groupoid of a log symplectic manifold. The third is the adjoint Lie groupoid integrating the tangent algebroid of a Riemann surface twisted by a divisor.

The second method is a gluing construction, whereby Lie groupoids defined on the open sets of an appropriate cover may be combined to obtain global integrations. This allows us to construct and classify the Lie groupoids integrating the given Lie algebroid. We apply this method to the aforementioned cases, albeit with small differences, and characterize the category of integrations in each case.

Dedication

Dedicated to my parents, Wenzhen and Ying.

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Contents

Chapter 1

Introduction

The notion of Lie algebroids and Lie groupoids goes back to Pradines [33], who observed that the infinitesimal object of a Lie groupoid is that of a Lie algebroid. A Lie groupoid, like a Lie group of which it naturally generalizes, measures a certain symmetry as explained in [43].

This field received renewed interest, when in the 80's, Weinstein [38] and Karasëv [21] observed that in many cases, there is a natural symplectic realization $\mathcal{G}$ for a given Poisson manifold M which has a Lie groupoid structure over M. The Lie algebroid of the symplectic groupoid $\mathcal{G} \rightrightarrows M$ is the natural Lie algebroid structure on the cotangent bundle T^*M induced by the Poisson structure. For this reason, the natural symplectic realization is called a symplectic groupoid. In particular, the source map of the symplectic groupoid is a surjective Poisson submersion. One hopes to replace the study of a Poisson manifold by its symplectic groupoid.

In his original paper [38], Weinstein observed that a Poisson manifold M, and more generally, a Lie algebroid, may fail to integrate to a smooth Lie groupoid. Since that time, a lot of effort has been made to come up increasingly sophisticated examples and to address the question of existence of integrations of Lie algebroids. The symplectic groupoid of the KirillovKostantSouriau Poisson structure on the dual of a Lie algebra was studied in [5]. This is a special case of Poisson Lie groups, whose was studied in [24, 23]. In [4], Cattaneo and Felder described the symplectic groupoid as an infinite–dimensional symplectic quotient. Using this idea, Crainic and Fernandes gave a sufficient and necessary condition integrability of Lie algebroids [6] and Poisson manifolds [7].

Since the concrete understanding of the topology of symplectic groupoids is useful in many applications, e.g. geometric quantization [18], and since explicit examples of symplectic groupoids are not very numerous, in this thesis, we give several geometric constructions of symplectic groupoids, and of Lie groupoids in general.

To a large extent, this thesis is based on and extends the contents of [12], and it is organized as follows:

In §2, we recall the notion of Lie groupoids and Lie algebroids, and some general constructions, following the terminology of [25]. We also recall the notion of projective blow-up.

In §3, we systematically apply the projective blow-up operation to Poisson manifolds, Lie groupoids and Poisson groupoids, and study how it affects the corresponding Lie algebroid and Lie bialgebroids. This work is inspired by the work of Weinstein [40, 41], Mazzeo–Melrose [28] and Monthubert [30].

In §4, we study the log symplectic manifolds, which are generically nondegenerate Poisson manifolds that drop rank along a smooth hypersurface. Through slightly different means, we reproduce the results of Guillemin–Miranda–Pires [15, 16], which describe the Poisson geometry near the degeneracy locus. We also reproduce the results of Radko [34], which completely classify the log symplectic structures on an orientable surface. We extend her results to the case of non-orientable surfaces.

In §5, we apply the birational construction in §2 to several different cases. The first example is the Lie algebroid of vector fields tangent to a collection of normal crossing hypersurfaces. The birational construction in this particular case is due to Monthubert [30]. The second example is the proper log symplectic manifold. The existence of the symplectic groupoid of a log symplectic manifold has been known since the work of Debord [8]. However, we gave a concrete geometric construction of the adjoint symplectic groupoid in terms of blow-up. The third example is the tangent algebroid of a Riemann surface twisted by a divisor. The representation of the Lie algebroid and its integration is the subject of Riemann-Hilbert correspondence and Stokes' phenomenon. This is carefully studied in [13].

In §6, inspired by the work of Nistor [31], we give a gluing construction of Lie groupoids over an open cover of the base manifold adapted to the orbits. This allows us to classify the integrations of certain Lie algebroids in a combinatorial fashion. The first example is the integrations of the Lie algebroid of vector fields tangent to a smooth hypersurface. The second example is the Hausdorff symplectic groupoids of a log symplectic manifold. The third example is the tangent algebroid of a Riemann surface twisted by a divisor.

Chapter 2

Preliminaries

In §2.1, we recall the notions of Lie groupoids and Lie algebroids, in the real smooth setting. The analogous notions in the complex holomorphic setting are mentioned without details. [25, 29] As an important case of Lie groupoids and Lie algebroids, we recall the notion of a symplectic groupoid of a Poisson manifold in §2.2. [10, 37]

In this thesis, the base manifold of a Lie groupoid, or the manifold of objects, is assumed to be a Hausdorff and second-countable manifold, real or complex. On the other hand, the space of arrows of Lie groupoid is **not** assumed to be Hausdorff, but satisfy the other axioms of a manifold.

In §2.3, we recall the notion of blow-up in the real smooth setting.

2.1 Lie theory

2.1.1 Lie groupoids

We begin with the definition of a set groupoid.

Definition 2.1.1. A ***groupoid*** is a small category such that the arrows are invertible.

As with the convention, we denote a groupoid by $(\mathcal{G}, M, s, t, m, \mathsf{id}, i)$, or $\mathcal{G} \rightrightarrows M$, where

(i) $\mathcal{G}$ is the set of arrows and M is the set of objects;

(ii) $s : \mathcal{G} \to M$ gives the source of an arrow and $t : \mathcal{G} \to M$ gives the target;

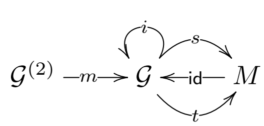
(iii) $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is the multiplication map, defined on the composable morphisms

$$\mathcal{G}^{(2)} := \mathcal{G}_s \times_t \mathcal{G} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\};$$

(iv) $\mathsf{id} : M \to \mathcal{G}$ gives the identity arrows;

(v) $i : \mathcal{G} \to \mathcal{G}, \quad g \mapsto g^{-1}$ gives the inverse.

The groupoid structure maps may be displayed as follows:



satisfying the expected compatibility conditions.

Definition 2.1.2. A (real) ***Lie groupoid*** is a groupoid $(\mathcal{G}, M, s, t, m, \mathsf{id})$ such that

(i) M is a Hausdorff, second-countable smooth manifold;

(ii) $\mathcal{G}$ is a second-countable smooth manifold, not necessarily Hausdorff;

(iii) the source s and the target t are surjective submersions;

(iv) the multiplication m and the identity id are smooth.

Remark 2.1.3. (i) The relaxation that $\mathcal{G}$ may be non-Hausdorff is customary in the literature (e.g. [6, 29]).

(ii) The inversion $i : \mathcal{G} \to \mathcal{G}$ is a diffeomorphism.

(iii) For a Lie groupoid $\mathcal{G} \rightrightarrows M$, the source fiber of a point $x \in M$ is the preimage $s^{-1}(x)$.

(iv) A Lie groupoid $\mathcal{G} \rightrightarrows M$ is ***étale*** if $\dim \mathcal{G} = \dim M$.

Example 2.1.4. For a connected manifold M, the ***pair groupoid*** of M is the groupoid $\mathrm{Pair}(M) \rightrightarrows M$ where $\mathrm{Pair}(M) = M \times M$, and the source and the target are the first and second projection.

The ***fundamental groupoid*** of M is the groupoid $\Pi_1 M \rightrightarrows M$ where $\Pi_1 M$ is the homotopy classes of paths and the source and the target are the initial point and the end point of the path.

Definition 2.1.5. For two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows L$, a ***Lie groupoid morphism*** from $\mathcal{G}$ to $\mathcal{H}$ is a smooth map of pairs $\Phi : (\mathcal{G}, M) \to (\mathcal{H}, L)$ compatible with the structure maps.

We say Φ is ***base preserving***, if $\Phi : M \to L$ is an isomorphism.

Example 2.1.6. Let M be a smooth manifold. The product of the source and the target from the fundamental groupoid to the pair groupoid, $(s, t) : \Pi_1 M \to \mathrm{Pair}(M)$, is a base preserving Lie groupoid morphism.

For points on the base $x, y \in M$, we denote the space of arrows from x to y, $s^{-1}(x) \cap t^{-1}(y)$ by $\mathcal{G}(x, y)$. In analogy with group actions, $\mathcal{G}_x = \mathcal{G}(x, x)$ is called the ***isotropy group*** at x. The equivalence relation on the base manifold M given by

$$x \sim y \Leftrightarrow y \in t(s^{-1}(x))$$

partitions M into equivalence classes called ***orbits*** of the groupoid.

Example 2.1.7. Let M be a connected smooth manifold. The isotropy group $\Pi_1(M)(x, x)$ is the fundamental group $\pi_1(M, x)$. For $\Pi_1(M) \rightrightarrows M$, there is only one orbit, namely M.

A set groupoid may be pulled back by a map as follows.

Definition 2.1.8. Let $\mathcal{H} \rightrightarrows L$ be a groupoid, not necessarily Lie, and let $f : M \to L$ be a map. The ***pullback groupoid*** $f^! \mathcal{H} \rightrightarrows M$ is the groupoid defined as follows:

1. $f^! \mathcal{H} = M\, {}_f\!\times_s \mathcal{H}\, {}_t\!\times_f M = \{(x, h, y) \in M \times G \times M \mid f(x) = s(h),\ t(h) = f(y)\}$.

2. The groupoid structure of $f^!\mathcal{H} \rightrightarrows M$ are as follows:

$$s'(x,h,y) = x, \quad t'(x,h,y) = y, \quad m'\left((x,h,y), \quad (y,k,z)\right) = (x, m(h,k), z),$$
$$\mathsf{id}'(x) = (x, \mathsf{id}\left(f(x)\right)), \quad (x,h,y)^{-1} = (y, h^{-1}, x).$$

□

In general, the pullback groupoid of a Lie groupoid by a smooth map is not a Lie groupoid. Here is a transversality condition that ensures the pullback groupoid is Lie.

Proposition 2.1.9. *[25] Let $\mathcal{H} \rightrightarrows L$ be a Lie groupoid, and let $f : M \to L$ be a smooth map. If $(s,t) : \mathcal{H} \rightrightarrows L \times L$ and $(f,f) : M \times M \to L \times L$ are transverse, and the target map of the pullback groupoid $t' : f^!\mathcal{H} \to M$ is a surjective submersion, then the pullback groupoid $f^!\mathcal{H} \rightrightarrows M$ is a Lie groupoid such that the map of pairs $(f^!, f) : (f^!\mathcal{H}, M) \to (\mathcal{H}, L)$ is a Lie groupoid morphism.*

Another useful notion in Lie groupoid theory is the action of a Lie groupoid.

Definition 2.1.10. An ***action of a Lie groupoid*** $\mathcal{H} \rightrightarrows L$ on a smooth map $f : M \to L$ is a smooth map

$$\rho : \mathcal{H}_s\times_f M \to M, \quad (h,x) \mapsto h \cdot x \tag{2.1.1}$$

such that

1. $f(h \cdot x) = t(h)$, for $(h,x) \in \mathcal{H}_s\times_f M$;
2. $h \cdot (g \cdot x) = (m(h,g)) \cdot x$, for $(h,g,x) \in \mathcal{H}_s\times_t \mathcal{H}_s\times_f M$;
3. $\mathsf{id}(f(x)) \cdot x = x$, for $x \in M$.

The action groupoid $\mathcal{H} \ltimes_\rho M \rightrightarrows M$ is diffeomorphic to $\mathcal{H}_s\times_f M$ with the following structure maps:

$$s(h,x) = x, \quad t(h,x) = h \cdot x,$$
$$m\left((g,x),(h, g\cdot x)\right) = (m(g,h), x), \quad \mathsf{id}(x) = (\mathsf{id}(f(x)), x).$$

□

Example 2.1.11. Let a Lie group G acts on a manifold M. In the groupoid language, this means the Lie group G acts on the trivial map $f : M \to \mathtt{pt}$. The action groupoid $G \ltimes M \rightrightarrows M$ is diffeomorphic to $G \times M$.

Let $E \to M$ be a vector bundle. The ***general linear groupoid*** $\mathtt{GL}(E) \rightrightarrows M$ is diffeomorphic to

$$\mathtt{GL}(E) \simeq \{(x,y,\phi_{xy}) \mid x \in M,\ y \in M,\ \phi_{xy} : E_x \xrightarrow{\sim} E_y \text{ is an invertible linear transformation}\} \tag{2.1.2}$$

with the structure maps as follows:

$$s(x,y,\phi_{xy}) = x, \quad t(x,y,\phi_{xy}) = y,$$
$$m\left((x,y,\phi_{xy}),(y,z,\phi_{yz})\right) = (x,z,\phi_{yz}\circ\phi_{xy}), \quad \mathsf{id}(x) = (x,x,\mathsf{id}_{xx}).$$

An action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a vector bundle $p : E \to M$ is nothing but a Lie groupoid morphism $\rho : \mathcal{G} \to \mathtt{GL}(E)$.

Example 2.1.12. Let $p : P \to M$ be a principal G-bundle. Consider the diagonal action of G on $P \times P$ by $g(u, v) = (gu,\ gv)$. The ***gauge groupoid*** $\mathtt{Gauge}(P) \rightrightarrows M$ is diffeomorphic to $(P \times P)/G$ with the following structure maps

$$\begin{aligned}
&\forall\ u, v, v', w \in P \ \text{ s.t. } p(v) = p(v') = x \\
&s([u, v]) = p(v), \quad t([u, v]) = p(u), \quad m\left([u, v], [v', w]\right) = ([u, w]), \quad \mathsf{id}(x) = ([v, v]).
\end{aligned}$$

□

We now define Lie groupoid cohomology. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $p : E \to M$ be a vector bundle. Let

$$\rho : \mathcal{G}_s \times_p E \to E, \quad (g, u) \mapsto g \cdot u$$

be a action. We write

$$\mathcal{G}^{(n)} = \{(g_1, g_2, \ldots, g_n) \in \mathcal{G} \times \ldots \times \mathcal{G} \mid t(g_i) = s(g_{i+1}),\ 1 \leq i \leq n-1\} \tag{2.1.3}$$

with the map

$$\delta^n : \mathcal{G}^{(n)} \to M, \quad (g_1, g_2, \ldots, g_n) \mapsto t(g_n).$$

The set of ***smooth n-cochains*** of $\mathcal{G} \rightrightarrows M$ with values in E is

$$C^n(\mathcal{G}, E) \coloneqq \Gamma(\mathcal{G}^{(n)}, (\delta^n)^* E). \tag{2.1.4}$$

The graded family $C^\bullet(\mathcal{G}, E)$ can be turned into a cochain complex with the codifferentials

$$d^n : C^n(\mathcal{G}, E) \to C^{n+1}(\mathcal{G}, E) \tag{2.1.5}$$

defined by

$$\begin{aligned}
d^0(c)(g) &= g \cdot c(s(g))) - c(t(g)), \\
d^n(c)(g_1, g_2, \ldots, g_{n+1}) &= g_1 \cdot c(g_2, g_3, \ldots, g_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i (g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\
&\quad + (-1)^{n+1} c(g_1, g_2, \ldots, g_n).
\end{aligned}$$

It is easy to check that $d^{n+1} \circ d^n = 0$. The ***Lie groupoid cohomology*** of $\mathcal{G} \rightrightarrows M$ with coefficients in E is the cohomology of the cochain complex $(C^\bullet(\mathcal{G}, E), d^\bullet)$.

Definition 2.1.13. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $\mathcal{H} \rightrightarrows L$ be a Lie subgroupoid. The ***tangent prolongation*** of $\mathcal{G} \rightrightarrows M$, denoted by $T\mathcal{G} \rightrightarrows TM$, is the Lie groupoid obtained by applying the tangent functor to $\mathcal{G} \rightrightarrows M$. The ***linearization*** of $\mathcal{G} \rightrightarrows M$ around $\mathcal{H} \rightrightarrows L$, $N\mathcal{H} \rightrightarrows NL$, is the Lie groupoid obtained by applying the normal functor:

$$\begin{array}{ccccc} T\mathcal{H} & \longrightarrow & T\mathcal{G} & \longrightarrow & N\mathcal{H} \\ {\scriptstyle Ts}\downarrow\downarrow{\scriptstyle Tt} & & {\scriptstyle Ts}\downarrow\downarrow{\scriptstyle Tt} & & {\scriptstyle Ns}\downarrow\downarrow{\scriptstyle Nt} \\ TL & \longrightarrow & TM & \longrightarrow & NL \end{array} \tag{2.1.6}$$

2.1.2 Lie algebroids

Similar to the case of Lie groups and Lie algebras, the infinitesimal object of a Lie groupoid is a Lie algebroid, defined as follows.

Definition 2.1.14. A Lie algebroid $(A, M, [\cdot,\cdot], a)$, or A for short, is a vector bundle $A \to M$ with a Lie bracket $[\cdot,\cdot]$ on the sections $\Gamma(A)$ and a bundle map $a : A \to TM$, called the ***anchor***, preserving the Lie bracket and satisfying the Leibniz rule

$$[X, fY] = f[X, Y] + a(X)(f)Y.$$

Just like the Lie functor from Lie groups to Lie algebras, there is a Lie functor from Lie groupoids to Lie algebroids.

Proposition 2.1.15. *[33] For a Lie groupoid $\mathcal{G} \rightrightarrows M$, the vector bundle*

$$\mathtt{Lie}(\mathcal{G}) := \mathsf{id}^* \ker\left(Ts : T\mathcal{G} \to TM\right)$$

with the bracket of the left-invariant vector fields and the restriction of the derivative Tt to $\ker(Ts)$ is a Lie algebroid.

As a vector bundle, we may identify $\mathtt{Lie}(\mathcal{G})$ with the normal bundle of the identity bisection, $N\left(\mathsf{id}(M)\right)$. When $\mathtt{Lie}(\mathcal{G}) \cong A$, then we say the Lie groupoid $\mathcal{G} \rightrightarrows M$ ***integrates*** the Lie algebroid A, or $\mathcal{G} \rightrightarrows M$ is an ***integration*** of A. A Lie algebroid A over M is ***integrable***, if there exists a Lie groupoid $\mathcal{G} \rightrightarrows M$ that integrates A.

Example 2.1.16. Let M be a manifold. Then $\mathtt{Lie}(\Pi_1(M)) \cong \mathtt{Lie}(\mathrm{Pair}(M)) \cong TM$.

Unlike Lie groupoids, it is not so easy to define a general Lie algebroid morphism. We will do it in steps. First we define a base-preserving Lie algebroid morphism.

Definition 2.1.17. For two Lie algebroids $(A, [\cdot,\cdot]_A, a_A)$ and $(B, [\cdot,\cdot]_B, a_B)$ over the same base manifold M, a ***base preserving Lie algebroid morphism*** from A to B is a bundle $\phi : A \to B$ compatible with the anchor map and the preserves the bracket. That is, for $X, Y \in \Gamma(A)$, we have

$$a_B \circ \phi(X) = a_A(X), \qquad \phi\left([X, Y]_A\right) = [\phi(X), \phi(Y)]_B.$$

For a base-preserving Lie groupoid homomorphism $\Psi : \mathcal{G} \to \mathcal{G}'$, we have the induced morphism of Lie algebroids

$$\mathtt{Lie}(\Psi) = T\Psi|_{\mathtt{Lie}(\mathcal{G})} : \mathtt{Lie}(\mathcal{G}) \to \mathtt{Lie}(\mathcal{G}').$$

Next, we describe the pullback of a Lie algebroid.

Proposition 2.1.18. *[25] Let (B, L, b) be a Lie algebroid, and let $f : M \to L$ be a smooth map. If the maps $f^*(b) : f^*(B) \to f^*(TL)$ and $Tf : TM \to f^*(TL)$ are transverse, then the fiber product of vector bundle*

$$A = f^! B = f^*(B) \oplus_{f^*(TL)} TM$$

with the following structure

(i) the anchor $a : A \to TM$ is the natural projection to TM;

(ii) for $(fX, x), (gY, y) \in \Gamma(A)$ where $f, g \in C^\infty(M)$, $X, Y \in \Gamma(B)$ and $x, y \in \Gamma(TM)$, we have

$$[(fX, x), (gY, y)] = (fg[X, Y] + x(g)Y - y(f)X, [x, y]),$$

is a Lie algebroid over M, called the ***pullback Lie algebroid*** *of B by f.*

If $\mathcal{H} \rightrightarrows L$ is a Lie groupoid, and $f : M \to L$ is a smooth map such that the pullback Lie groupoid $f^!\mathcal{H} \rightrightarrows M$ exists, then the pullback Lie algebroid $f^!(\texttt{Lie}(\mathcal{H}))$ exists and we have $\texttt{Lie}(f^!\mathcal{H}) = f^!(\texttt{Lie}(\mathcal{H}))$. Now, we are ready to define a general Lie algebroid morphism.

Definition 2.1.19. Let (A, M, a) and (B, L, b) be a Lie algebroids. A Lie algebroid morphism is a bundle map $(\phi, f) : (A, M) \to (B, L)$

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & L \end{array} \tag{2.1.7}$$

such that

(i) $b \circ \phi = Tf \circ a$;

(ii) the pullback Lie algebroid $f^!B$ exists, and the induced bundle map

$$\phi^! : A \to f^!B, \quad X \mapsto a(X) \oplus \phi(X)$$

is a base preserving Lie algebroid morphism over M.

As expected, if (Φ, f) is a Lie groupoid morphism from $\mathcal{G} \rightrightarrows M$ to $\mathcal{H} \rightrightarrows L$, then $(\texttt{Lie}(\Phi), f)$ is a Lie algebroid morphism from $\texttt{Lie}(\mathcal{G})$ to $\texttt{Lie}(\mathcal{H})$.

Definition 2.1.20. Let L be a closed embedded submanifold of M. Then (B, L, b) is a ***Lie subalgebroid*** of (A, M, a) if and only if B is a subbundle of $A|_L$, and the bundle inclusion is a Lie algebroid morphism.

Equivalently, it may be phrased explicitly as

(i) the anchor $a : A \to TM$ restricted to L yields $b : B \to TL$;

(ii) if $X, Y \in \Gamma(A)$ have $X|_L, Y|_L \in \Gamma(B)$, then $[X, Y]|_L \in \Gamma(B)$;

(iii) if $X, Y \in \Gamma(A)$ have $X|_L = 0$ and $Y|_L \in \Gamma(B)$, then $[X, Y]|_L = 0$.

Remark 2.1.21. In Definition 2.1.20, if L is a closed hypersurface, then (iii) may be dropped for the following reason.

Let f be a defining function of L, i.e. $f(x) = 0$ for $x \in L$, $f(x) \neq 0$ for $x \notin L$, and $df|_L \neq 0$. For any $X \in \Gamma A$ such that $X|_L = 0$, we may write $X = fX'$ for some $X' \in \Gamma A$. For $Y \in \Gamma A$ such that $Y|_L \in \Gamma B$, the vector field $a(Y)$ is tangent to L. It follows that $a(Y)(f)|_L = 0$. Hence, the Leibniz rule

$$[fX', Y] = -a(Y)(f)X' + f[X', Y] \tag{2.1.8}$$

shows that $[X, Y]|_L = 0$.

That is, for a Lie algebroid A over M and a closed hypersurface $L \subset M$, a subbundle B of $A|_L$ is a Lie subalgebroid if and only if B itself is a Lie algebroid.

Let (A, M, a) be a Lie algebroid, and let $p : E \to M$ be a vector bundle. An ***A-connection*** on E is a C^∞-linear morphism

$$\nabla : \Gamma(E) \to \Gamma(A^*) \otimes_{C^\infty(M)} \Gamma(E) \tag{2.1.9}$$

satisfying the Leibniz rule $\nabla(fu) = d_A(f)u + f\nabla u$ for all $f \in C^\infty(M)$ and $u \in \Gamma(E)$, where $d_A = a \circ d$. If

$$\nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X \tag{2.1.10}$$

for all $X, Y \in \Gamma(A)$, then ∇ is said to be flat, and (E, ∇) is called an ***A-module***, or equivalently we say (E, ∇) is a ***representation*** of A. □

The set of ***smooth n-cochains*** of A with values in E is

$$C^n(A, E) \coloneqq \Gamma(\wedge^n A^* \otimes E). \tag{2.1.11}$$

The graded family $C^\bullet(A, E)$ can be turned into a cochain complex with the codifferentials

$$\begin{aligned} d^0\xi(X) &= \nabla_X \xi, \\ d^n\omega(X_1, X_2, \ldots, X_{n+1}) &= \sum_{i<j} (-1)^{i+j-1} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i \nabla_{X_i} \omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{n+1}). \end{aligned}$$

It is easy to check that $d^{n+1} \circ d^n = 0$. The ***Lie algebroid cohomology*** of A with coefficients in E is the cohomology of the cochain complex $(C^\bullet(A, E), d^\bullet)$.

The relation between Lie groupoid cohomology and Lie algebroid cohomology is explained in [1].

Example 2.1.22. Let $P \to M$ be a principal G-bundle. The ***Atiyah algebroid*** $\mathtt{At}(P)$ of P is the Lie algebroid of the gauge groupoid $\mathtt{Gauge}(P) \rightrightarrows P$, which fits in the following short exact sequence:

$$0 \longrightarrow P \times_G \mathfrak{g} \longrightarrow \mathtt{At}(P) \longrightarrow TM \longrightarrow 0 \tag{2.1.12}$$

where $P \times_G \mathfrak{g}$ is the associated bundle. □

We may replace the notions of Lie groupoids and Lie algebroids in the smooth category with the analogous ones in the holomorphic category.

2.1.3 The category of integrations

In this subsection, we closely follow [12], somewhat expanding the exposition.

A finite-dimensional Lie algebra $\mathfrak{g}$ determines a lattice $\Lambda(\mathfrak{g})$ of connected Lie groups which integrate it: G' covers G in $\Lambda(\mathfrak{g})$ if there is a morphism $G' \to G$ inducing the identity map on $\mathfrak{g}$. The initial object of this lattice is the simply-connected integration $\widetilde{G}$; all other groups in the lattice are quotients of $\widetilde{G}$ by discrete subgroups of its center; and the terminal object, when it exists, is called the *adjoint form* of the group.

When a Lie algebroid is integrable, its lattice, or more properly, its category of integrating Lie groupoids has similar properties to those described above, but with some important differences. For instance, there is the question of which integrations are Hausdorff. Also, unlike the case of Lie groups, where morphisms among integrations are covering maps, for Lie groupoids these morphisms are only local diffeomorphisms, which may fail to be covering maps globally.

We now define the category of integrations of a Lie algebroid; for this we need the groupoid analog of (simple-)connectedness.

Definition 2.1.23. Any Lie groupoid $\mathcal{G} \rightrightarrows M$ has a well-defined subgroupoid $\mathcal{G}^c \rightrightarrows M$ all of whose source fibers are connected. If $\mathcal{G} = \mathcal{G}^c$, we say that $\mathcal{G}$ is ***source-connected***.

If the source fibres of a source-connected groupoid are also simply connected, then the groupoid is called ***source-simply-connected***, or ***ssc*** for short.

By a result of Moerdijk-Mrčun [29], an integrable Lie algebroid A has a source-simply-connected integration $\mathcal{G}^{ssc}$ which is unique up to a canonical isomorphism. Their results also show that any source-connected integration $\mathcal{G}$ of A receives a unique morphism

$$p : \mathcal{G}^{ssc} \to \mathcal{G}, \tag{2.1.13}$$

which is a surjective local diffeomorphism and a covering map along the source fibers. As a result, $\mathcal{G}^{ssc}$ may be viewed as the initial object of a category of integrations, which we now define.

Definition 2.1.24. To a Lie algebroid A over M, we associate two categories

$$\mathtt{Gpd}^{\mathcal{H}}(A) \subset \mathtt{Gpd}(A) :$$

1. Objects of $\mathtt{Gpd}(A)$ are pairs $(\mathcal{G}, \phi)$, where $\mathcal{G}$ is a source-connected Lie groupoid over M and $\phi : \mathtt{Lie}(\mathcal{G}) \to A$ is an isomorphism covering the identity on M;
2. A morphism from $(\mathcal{G}, \phi)$ to $(\mathcal{G}', \phi')$ is a Lie groupoid morphism $\psi : \mathcal{G} \to \mathcal{G}'$ such that $\phi = \phi' \circ \mathtt{Lie}(\psi)$, where $\mathtt{Lie}(\psi)$ is the Lie algebroid morphism induced by ψ.

The subcategory of Hausdorff integrations is then denoted by $\mathtt{Gpd}^{\mathcal{H}}(A)$. A terminal object, when it exists, is called the ***adjoint*** groupoid $\mathcal{G}^{adj}$. There is at most one morphism between any two objects.

In analogy with Lie groups, each of the integrations in $\mathtt{Gpd}(A)$ may be described as a quotient of the source-simply-connected integration. To describe Lie groupoid quotients, we recall that a Lie groupoid $\mathcal{G} \rightrightarrows M$ is ***étale*** when $\dim(\mathcal{G}) = \dim(M)$; a subgroupoid $\mathcal{N} \rightrightarrows L$ of $\mathcal{G} \rightrightarrows M$ is ***wide*** when $L = M$; such a wide subgroupoid is ***normal*** if for all $g \in \mathcal{G}(x, y) = s^{-1}(x) \cap t^{-1}(y)$, we have

$$g\mathcal{N}_x g^{-1} = \mathcal{N}_y$$

where $\mathcal{N}_x = \mathcal{N}(x,x)$ is the isotropy group of $\mathcal{N}$ at $x \in M$. A normal subgroupoid defines an equivalence relation $R \subset \mathcal{G} \times \mathcal{G}$ via

$$R \coloneqq \{(g,g') \in \mathcal{G} \times \mathcal{G} \mid \mathcal{N}g = \mathcal{N}g'\}, \tag{2.1.14}$$

whose equivalence classes are the right cosets $\mathcal{G}/\mathcal{N}$. Finally, $\mathcal{N}$ is ***totally disconnected*** if $\mathcal{G}(x,y) = \varnothing$ for $x \neq y$.

We now describe the quotient construction for Lie groupoids, following [19, Theorem 3.3], which treats Hausdorff Lie groupoids but is easily extended to the general case:

Theorem 2.1.25 (Higgins-Mackenzie [19])**.** *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\mathcal{N} \rightrightarrows M$ a totally disconnected normal Lie subgroupoid. Then the quotient $\mathcal{G}/\mathcal{N} \rightrightarrows M$ is a Lie groupoid; it integrates* $\mathtt{Lie}(\mathcal{G})$ *if and only if $\mathcal{N} \rightrightarrows M$ is étale. Furthemore, if $\mathcal{G}$ is Hausdorff, then the quotient groupoid is Hausdorff if and only if $\mathcal{N}$ is closed in $\mathcal{G}$.*

Sketch of proof: The only difference with the proof in [19] is that to obtain smoothness of $\mathcal{G}/\mathcal{N}$, we have to use a theorem of Godement [36, Theorem II.3.12.2], which states that if X is a possibly non-Hausdorff smooth manifold, and $R \subset X \times X$ is an equivalence relation, then X/R is a possibly non-Hausdorff smooth manifold if and only if R is a wide Lie subgroupoid of the pair groupoid $X \times X$. Furthermore, if X is Hausdorff, then X/R is Hausdorff if and only if $R \subset X \times X$ is closed. □

The following equivalence result follows immediately from Theorem 2.1.25.

Theorem 2.1.26. *A ssc integration $\mathcal{G}^{ssc}$ of the Lie algebroid A defines an equivalence of categories between $\mathtt{Gpd}(A)$ and the poset $\Lambda(\mathcal{G}^{ssc})$ of étale, totally disconnected, normal Lie subgroupoids of $\mathcal{G}^{ssc}$, via*

$$\mathbf{N} : \mathtt{Gpd}(A) \to \Lambda(\mathcal{G}^{ssc}),$$

which takes a groupoid $\mathcal{G}$ to the kernel of the canonical morphism $p : \mathcal{G}^{ssc} \to \mathcal{G}$, and

$$\mathbf{G} : \Lambda(\mathcal{G}^{ssc}) \to \mathtt{Gpd}(A),$$

which takes the normal subgroupoid $\mathcal{N} \subset \mathcal{G}^{ssc}$ to the quotient groupoid $\mathcal{G}^{ssc}/\mathcal{N}$.

In the case that $\mathcal{G}^{ssc}$ is itself Hausdorff, then the equivalence identifies $\mathtt{Gpd}^{\mathcal{H}}(A)$ with the subposet $\Lambda^{\mathcal{H}}(A) \subset \Lambda(A)$ of closed subgroupoids.

Example 2.1.27. The tangent Lie algebroid TM of a connected manifold M has ssc integration given by the fundamental groupoid $\Pi_1(M)$, and has adjoint groupoid given by the pair groupoid $\mathrm{Pair}(M) = M \times M$. We determine all integrations of TM as follows. By Theorem 2.1.26, any integration $\mathcal{G}$ of TM may be described as a quotient of $\Pi_1(M)$ by a étale, totally disconnected normal Lie subgroupoid $\mathcal{N}$.

Since $\mathcal{N}$ is totally disconnected, it is contained in the isotropy subgroupoid of $\Pi_1(M)$, which is étale, so that $\mathcal{N}$ is automatically étale. The fact that $\mathcal{N}$ is normal implies that $\mathcal{N}$ is determined by its intersection with the isotropy group at any point $x_0 \in M$, which is simply the fundamental group $\Pi_1(x_0,x_0) = \pi_1(M,x_0)$ based at x_0. Hence $\mathcal{N}$ is uniquely determined by the choice of a normal subgroup of the fundamental group of M, and so the category of integrations is equivalent to the lattice of normal subgroups of the fundamental group of M:

$$\mathtt{Gpd}^{\mathcal{H}}(TM) \cong \mathtt{Gpd}(TM) \cong \Lambda(\pi_1(M)).$$

The integrations of TM are all Hausdorff, since the normal subgroupoids described above are closed in $\Pi_1(M)$.

2.2 Poisson manifolds and symplectic groupoids

2.2.1 Poisson manifolds

Definition 2.2.1. A ***Poisson manifold*** is a smooth manifold M with a Lie bracket

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

satisfying the Leibniz rule

$$\{fg, h\} = f\{g,h\} + g\{f,h\}.$$

For a function $f \in C^\infty(M)$, the ***Hamiltonian vector field*** of f, denoted by X_f, is the vector field defined by the derivation $\{f,\cdot\} : C^\infty(M) \to C^\infty(M)$.

Let $\mathfrak{X}(M) = \Gamma(TM)$ be the space of vector fields on M, and let $\mathfrak{X}^k(M) = \Gamma(\wedge^k TM)$ be the k-vector fields. The graded space of multi-vector fields

$$\mathfrak{X}^\bullet(M) = \bigoplus_k \mathfrak{X}^k(M)$$

carries the Schouten bracket [35]

$$[\cdot,\cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \longrightarrow \mathfrak{X}^{k+l-1}(M)$$

uniquely extending the Lie bracket on vector fields, making $\mathfrak{X}^\bullet(M)$ into a graded Lie algebra such that the bracket is also a derivation, i.e. for $X \in \mathfrak{X}^k(M)$ and $Y \in \mathfrak{X}^l(M)$ and $Z \in \mathfrak{X}^\bullet(M)$, we have

(i) $[X,Y] = -(-1)^{(k-1)(l-1)}[Y,X]$;

(ii) $[X, Y\wedge Z] = [X,Y]\wedge Z + (-1)^{(k-1)l} Y \wedge [X,Z]$;

(iii) $[X,[Y,Z]] = [[X,Y],Z] + (-1)^{(k-1)(l-1)}[Y,[X,Z]]$.

Since $\{f,g\}$ only depends on df and dg, there exists a bi-vector field $\pi \in \mathfrak{X}^2(M)$ such that

$$\{f,g\} = \pi(df, dg). \tag{2.2.1}$$

Abusing the notation, let π also be the associated skew-symmetric bundle map

$$\pi : T^*M \to TM, \quad \alpha \mapsto \iota_\alpha \pi.$$

Note we have $\pi(df) = X_f$, and the Poisson bracket $\{\cdot,\cdot\}$ in (2.2.1) satisfies the Jacobi identity [22] if and only if

$$[\pi,\pi] = 0.$$

Consequently, we may write the Poisson manifold $(M, \{\cdot,\cdot\})$ equivalently as (M,π).

Definition 2.2.2. A smooth map between Poisson manifolds

$$\phi : (M, \{\cdot,\cdot\}_M) \longrightarrow (L, \{\cdot,\cdot\}_L)$$

is a Poisson morphism if and only if $\phi^* : C^\infty(L) \to C^\infty(M)$ is a Lie algebra morphism.

Equivalently,

$$\phi : (M, \pi) \longrightarrow (L, \sigma)$$

is Poisson if and only if ϕ preserves the Poisson bi-vector, i.e. $\phi_*(\pi) = \sigma$.

For a Poisson manifold (M, π), the cotangent bundle T^*M with the anchor $\pi : T^*M \to TM$ and the ***Koszul bracket***

$$[\alpha, \beta] = L_{\pi(\alpha)}\beta - L_{\pi(\beta)}\alpha - d\pi(\alpha, \beta),$$

is a Lie algebroid. We call it the ***Poisson cotangent Lie algebroid*** of (M, π), or the ***Poisson algebroid*** for short, and denote it by $T^*_\pi M$.

For a Poisson manifold (M, π), the graded complex $\mathfrak{X}^\bullet(M)$carries a natural codifferential

$$d_\pi : \mathfrak{X}^n(M) \to \mathfrak{X}^{n+1}(M), \quad X \mapsto [\pi, X] \tag{2.2.2}$$

It is easy to verify that $[\pi, \pi] = 0$ implies $d_\pi \circ d_\pi = 0$. The ***Poisson cohomology*** of (M, π), denoted by $H^\bullet_\pi(M)$, is defined to be the cohomology of the complex $(\mathfrak{X}^\bullet(M), d_\pi)$, which is nothing but the Lie algebroid cohomology of the Poisson algebroid $T^*_\pi M$.

The zeroth Poisson cohomology H^0_π is the Casimirs; the first Poisson cohomology H^1_π is the Poisson vector fields mod the Hamiltonians vector fields; the second Poisson cohomology H^2_π is the infinitesimal deformations of the Poisson structure.

When π is non-degenerate, the anchor $\pi^\sharp$ extends to an isomorphism between $(\mathfrak{X}^\bullet(M), d_\pi)$ and the deRham complex $(\Omega^\bullet(M), d_{\mathtt{dR}})$. Similar to the deRham cohomology, the Poisson cohomology may be computed inductively using the Mayer-Vietoris sequence of an open cover.

2.2.2 Symplectic groupoids

A Lie bialgebroid is the infinitesimal object of a so-called Poisson groupoid. Among the Poisson groupoids, the so-called symplectic groupoids are of particular interest.

Definition 2.2.3. [21, 38] A ***Poisson groupoid*** is a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a multiplicative Poisson structure σ. That is, the Poisson structure σ is on $\mathcal{G}$, and the graph of multiplication

$$\mathtt{Graph}(m) = \{(g, h, m(g, h)) \mid (g, h) \in \mathcal{G}_t \times_s \mathcal{G}\}$$

is coisotropic inside $(\mathcal{G} \times \mathcal{G} \times \mathcal{G}, \sigma \oplus \sigma \oplus -\sigma)$.

A ***symplectic groupoid*** is a Poisson groupoid $(\mathcal{G} \rightrightarrows M, \sigma)$ such that σ is non-degenerate.

Remark 2.2.4. (i) Let $(\mathcal{G} \rightrightarrows M, \sigma)$ be a Poisson groupoid.

(1) The induced bivector $\pi = s_*(\sigma) = -t_*(\sigma)$ on M is Poisson.

(2) The identity bisection $\mathsf{id}(M)$ is coisotropic.

(ii) Let $(\mathcal{G} \rightrightarrows M, \sigma)$ be a symplectic groupoid. Let $\omega = \sigma^{-1}$ be the corresponding symplectic structure, i.e. $\omega^\sharp : TM \to T^*M$ is the inverse of $\sigma^\sharp : T^*M \to TM$.

(1) The identity bisection $\mathsf{id}(M)$ is Lagrangian inside $(\mathcal{G}, \omega)$.

(2) We have

$$m^*(\omega) = s^*\omega + t^*\omega,$$

and the graph of multiplication $\mathtt{Gr}_m$ is Lagrangian inside $(\mathcal{G} \times \mathcal{G} \times \mathcal{G}, \omega \oplus \omega \oplus -\omega)$.

For a Poisson groupoid $(\mathcal{G} \rightrightarrows M, \sigma)$, since the identity bisection $\mathsf{id}(M)$ is coisotropic, its conormal bundle $N^*(\mathsf{id}(M))$ inherits a Lie algebroid structure [39]. Now the Lie algebroid $A = \mathtt{Lie}(\mathcal{G})$ may be identified with $N(\mathsf{id}(M))$, so its dual A^* is naturally a Lie algebroid. In fact, (A, A^*) form a so-called Lie bialgebroid, which we now explain.

For a Lie algebroid $(A, M, [\cdot,\cdot], a)$, we define the A-differential, the Lie derivative and the contraction as follows:

1. The A-differential

$$\begin{aligned}
&d : \Gamma(\wedge^n A^*) \to \Gamma(\wedge^{n+1} A^*),\\
&d\phi(X_1, \dots, X_n) = \sum_{k=1}^{n+1} (-1)^{k+1} a(X_k)(\phi(X_1, \dots, \hat{X}_k, \dots, X_{n+1}))\\
&+ \sum_{j<k} (-1)^{j+k} \phi([X_j, X_k], X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{n+1});
\end{aligned} \tag{2.2.3}$$

2. the Lie derivative $\mathcal{L} : \Gamma(A \otimes \wedge^n A^*) \to \Gamma(\wedge^n A^*)$ is defined as

$$\mathcal{L}_X(\phi)(Y_1, \dots, Y_n) = a(X)(\phi(Y_1, \dots, X_n)) - \sum_{k=1}^{n} \phi(Y_1, \dots, [X, Y_k], \dots, Y_n); \tag{2.2.4}$$

3. the contraction $\iota : \Gamma(A \otimes \wedge^{n+1} A^*) \to \Gamma(\wedge^n A^*)$ is defined as

$$\iota_X(\phi)(Y_1, \dots, Y_n) = \phi(X, Y_1, \dots, Y_n). \tag{2.2.5}$$

The A-differential, the Lie derivative and the contraction satisfies the usual properties similar to those of the differential forms.

Definition 2.2.5. Let $(A, M, a, [\cdot,\cdot])$ be a Lie algebroid. If the dual bundle A^* carries a Lie algebroid structure $(A^*, M, a_*, [\cdot,\cdot]_*)$ such that

$$d_*[X, Y] = \mathcal{L}_X d_* Y - \mathcal{L}_Y d_* X, \tag{2.2.6}$$

then (A, A^*) is a ***Lie bialgebroid***.

Remark 2.2.6. For a Poisson manifold (M, π), the Poisson algebroid $T^*_\pi M$ and the tangent algebroid TM form a Lie bialgebroid [26].

Proposition 2.2.7. *[26] For a Poisson groupoid $(\mathcal{G} \rightrightarrows M, \sigma)$, the Lie algebroid $A = \mathtt{Lie}(\mathcal{G})$ and A^* form a Lie bialgebroid.*

We write $\texttt{Lie}(\mathcal{G},\sigma) = (A, A^*)$ to indicate that the Lie bialgebroid (A, A^*) is the infinitesimal object of the Poisson groupoid $(\mathcal{G} \rightrightarrows M, \sigma)$.

Example 2.2.8. Let (M,π) be a Poisson manifold. Then $(\mathrm{Pair}(M), \pi \oplus -\pi)$ is a Poisson groupoid over (M,π) and

$$\texttt{Lie}(\mathrm{Pair}(M), \pi \oplus -\pi) = (TM, T^*_\pi M).$$

Example 2.2.9. For a symplectic groupoid $(\mathcal{G} \rightrightarrows M, \sigma)$, let $\pi = s_*(\sigma)$ be the induced Poisson structure on M. Then,

$$\texttt{Lie}(\mathcal{G},\sigma) = (T^*_\pi M, TM).$$

Remark 2.2.10. [26] The converse to Example 2.2.9 is also true. That is, if $\texttt{Lie}(\mathcal{G},\sigma) = (T^*_\pi M, TM)$, then $(\mathcal{G} \rightrightarrows M, \sigma)$ is a symplectic groupoid of (M,π).

For a Poisson manifold (M,π), the ssc integration $\mathcal{G}^{ssc}$ of the Lie algebroid $T^*_\pi M$ carries a natural multiplicative symplectic structure σ, making $(\mathcal{G}^{ssc},\sigma)$ a symplectic groupoid for (M,π). In general, however, other integrations of $T^*_\pi M$ do not necessarily admit multiplicative symplectic structures. On the other hand, multiplicative symplectic forms behave well under pullbacks, in the following sense.

Proposition 2.2.11. *Let $\phi : \mathcal{G}' \to \mathcal{G}$ be a morphism between groupoids integrating $T^*_\pi M$. If $\sigma \in \Omega^2(\mathcal{G})$ is multiplicative and symplectic, then so is $\phi^*\sigma \in \Omega^2(\mathcal{G}')$.*

Remark 2.2.12. A immediate consequence of Proposition 2.2.11 is the following. For a Poisson manifold (M,π), if the adjoint integration of $T^*_\pi M$ admits a multiplicative symplectic structure, then all integrations of $T^*_\pi M$ are symplectic groupoids.

2.3 Projective blow-up

The notion of blow-up originates from algebraic geometry, e.g. [17], Prop. II.7.14. In this thesis, however, we work in the category of real smooth manifolds, or the category of complex analytic manifolds.

Let M be a real smooth manifold, and let L be a closed submanifold such that $\mathrm{codim}\,(L) \geq 2$. [1] We denote by $\mathsf{Bl}_L(M)$ the real projective blow-up of M along L. To construct $\mathsf{Bl}_L(M)$ from M, we replace L by the projectivization of its normal bundle, $\mathbb{P}(NL)$, which then defines a hypersurface $E \subset \mathsf{Bl}_L(M)$ called the exceptional divisor. The blow-down map

$$p : \mathsf{Bl}_L(M) \to M$$

is a diffeomorphism away from E and coincides with the bundle projection $\mathbb{P}(NL) \to L$ upon restriction to the exceptional divisor.

Example 2.3.1. For $X = \mathbb{R}^n$ and $Y = \{(x_1,\ldots,x_m,0,\ldots,0) \in X | x_i \in \mathbb{R}\} \subset X$, we have

$$\mathsf{Bl}_Y(X) \cong \{((x_1,\ldots,x_m),[y_{m+1} : y_{m+2} : \ldots : y_n]) \in \mathbb{R}^m \times \mathbb{RP}^{n-m-1} \mid x_i y_j = y_i x_j,\ m+1 \leq i,j \leq n\}.$$

[1] Alternatively in the complex analytic setting, M is a complex manifold, and L a closed complex submanifold.

Any submanifold $S \subset M$ having clean[2] intersection with L may be "pulled back" to $\mathsf{Bl}_L(M)$, by forming the ***proper transform*** (a.k.a the strict transform)

$$\overline{\overline{\overline{S}}} \coloneqq \overline{p^{-1}(S\backslash L)},$$

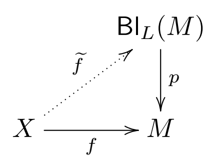
where the closure is taken in $\mathsf{Bl}_L(M)$. The proper transform $\overline{\overline{\overline{S}}}$ is itself a submanifold, naturally isomorphic to $\mathsf{Bl}_{L\cap S}(S)$. Of course, if $L \cap S$ has codimension 1 in S, the blowdown map restricts to a diffeomorphism $\overline{\overline{\overline{S}}} \to S$.

The notion of proper transform has a useful generalization: if a map $f : X \to M$ has clean intersection with L, then f lifts uniquely to a map from $\mathsf{Bl}_{f^{-1}(L)}(X)$ to $\mathsf{Bl}_L(M)$. We require a special case of this result, in which the domain remains unmodified.

Proposition 2.3.2. *([2], Theorem 4.4) Let $L \subset M$ be a closed submanifold of codimension ≥ 2. If $f : X \to M$ is a smooth map and $Y = f^{-1}(L) \subset X$ is a hypersurface such that the bundle map*

$$Nf : NY \to f^*NL$$

induced by the derivative Tf is injective, then there exists a unique smooth map $\widetilde{f} : X \to \mathsf{Bl}_L(M)$ such that $f = p \circ \widetilde{f}$.

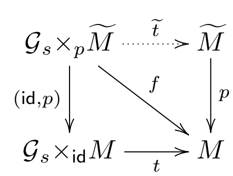


(2.3.1)

As a first application, we use Proposition 2.3.2 to construct the action groupoid over the blow-up of a $\mathcal{G}$-invariant submanifold.

Proposition 2.3.3. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $L \subset M$ be a closed $\mathcal{G}$–invariant submanifold, i.e. L is a union of $\mathcal{G}$-orbits. Let $\widetilde{M} = \mathsf{Bl}_L(M)$ be the blow-up of M along L.*

The target map $t : \mathcal{G} \to M$ may be interpreted as a Lie groupoid action $t : \mathcal{G}_s{\times}_{\mathsf{id}}M \to M$. This action lifts to an action $\widetilde{t} : \mathcal{G}_s{\times}_p\widetilde{M} \to \widetilde{M}$.



(2.3.2)

Proof. Let $E \subset \widetilde{M}$ be the exceptional divisor. The preimage of $L \subset M$ with respect to the map $f = t \circ (\mathsf{id}, p)$ is $\mathcal{G}_s{\times}_pE$, a hypersurface of $\mathcal{G}_s{\times}_p\widetilde{M}$. Since $Np : NE \to NL$ is an injective bundle map and L is $\mathcal{G}$-invariant, it follows that

$$Nf : N(\mathcal{G}_s{\times}_pE) \to NL$$

is an injective bundle morphism. By Proposition 2.3.2, we get a smooth map $\widetilde{t} : \mathcal{G}_s{\times}_p\widetilde{M} \to \widetilde{M}$, which is a groupoid action by continuity. □

[2]Submanifolds S, L have clean intersection when $S \cap L$ is a submanifold and $T(S \cap L) = TS \cap TL$.

Remark 2.3.4. The action groupoid $\mathcal{G} \ltimes_{\tilde{t}} \widetilde{M}$ is isomorphic to the blow-up of $\mathcal{G}$ along the subgroupoid $\mathcal{G}|_L$. The map

$$(\mathsf{id}, p) : \mathcal{G} \ltimes_{\tilde{t}} \widetilde{M} = \mathcal{G}_s \times_p \widetilde{M} \to \mathcal{G} = \mathcal{G} \times_M M$$

is the blow-down map, and together with $p : \widetilde{M} \to M$ is a groupoid morphism.

The restriction of the action groupoid $\mathcal{G} \ltimes_{\tilde{t}} \widetilde{M} \rightrightarrows \widetilde{M}$ to the exception divisor $E \simeq \mathbb{P}(NL) \subset \widetilde{M}$ is naturally isomorphic to the projectivization of the linearization of $\mathcal{G}|_L \rightrightarrows L$ (Definition 2.1.13). That is,

$$\left(\mathcal{G} \ltimes_{\tilde{t}} \widetilde{M}\right)|_E \rightrightarrows E \quad \simeq \quad \mathbb{P}\left(N(\mathcal{G}|_L)\right) \rightrightarrows \mathbb{P}(NL). \tag{2.3.3}$$

Chapter 3

Birational constructions

In this chapter, we roughly follow the presentation of [12], albeit with more detailed and improved perspectives. We also include some new examples.

In §3.1, we reproduce §2.3 of [12]. In §3.2, we describe the blow-up of Poisson groupoids and the elementary modification of Lie bialgebroids. In §3.3, we describe the fiber product constructions of Lie groupoids, generalizing the results in [30].

3.1 Blow-up of Lie groupoids

In this section, we demonstrate that the projective blow-up of a Lie groupoid $\mathcal{G} \rightrightarrows M$ along a subgroupoid $\mathcal{H} \rightrightarrows L$ inherits a Lie groupoid structure, once a certain degeneracy locus is removed. We restrict our attention to the case that the base L of the subgroupoid has codimension 1.

3.1.1 Lifting theorem

To lift the groupoid operations from $\mathcal{G}$ to the blow-up, we utilize the lifting criterion for smooth maps given in Proposition 2.3.2. The key point is the lifting of the groupoid multiplication; to apply the criterion here, we need the following description of the normal bundle of a fiber product of smooth maps of pairs. Recall that if $X \subset Y$ and $L \subset M$ are submanifolds, then $f : (Y, X) \to (M, L)$ is a smooth map of pairs when $f : Y \to M$ is a smooth map such that $f(X) \subset L$. Such a morphism of pairs induces a morphism of normal bundles

$$Nf : NX \to f^*NL,$$

defined to be the quotient of $(Tf)|_X : TY|_X \to f^*TM$ by $T(f|_X) : TX \to f^*TL$.

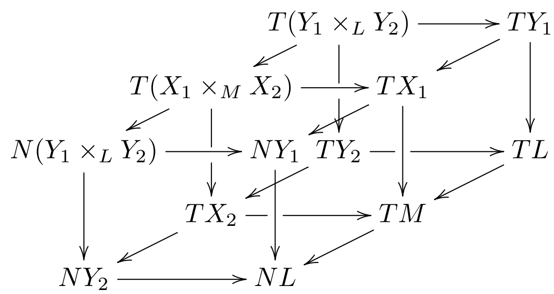
Lemma 3.1.1. *Let $f_1 : (Y_1, X_1) \to (M, L)$ and $f_2 : (Y_2, X_2) \to (M, L)$ be transverse smooth maps of pairs*[1]*. Then the fiber product of submanifolds $X_1 \times_L X_2 \subset Y_1 \times_M Y_2$ has normal bundle given by*

$$N(X_1 \times_L X_2) = NX_1 \times_{NL} NX_2.$$

Proof. Since $f_1 : X_1 \to M$ and $f_2 : X_2 \to M$ are transverse, their derivatives $Tf_1 : TX_1 \to TM$ and $Tf_2 : TX_2 \to TM$ are transverse. By the universal property of the fiber products, we have

[1] That is, both $f_1 : Y_1 \to M$, $f_2 : Y_2 \to M$ and $f_1|_{X_1} : X_1 \to L$, $f_2|_{X_2} : X_2 \to L$ are transverse.

$T(X_1 \times_M X_2) = TX_1 \times_{TM} TX_2$. Likewise, we have $T(Y_1 \times_L Y_2) = TY_1 \times_{TL} TY_2$.



(3.1.1)

That is, the second and third levels of Diagram 3.1.1 are fiber products. Using the universal property of fiber products again, the first level is also a fiber product. □

Theorem 3.1.2. *Let $\mathcal{H} \rightrightarrows L$ be a closed Lie subgroupoid of $\mathcal{G} \rightrightarrows M$ over the closed hypersurface L, and define*

$$[\mathcal{G}:\mathcal{H}] \coloneqq \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}) \backslash (\overline{\overline{s^{-1}(L)}} \cup \overline{\overline{t^{-1}(L)}}), \tag{3.1.2}$$

where s and t are the source and target maps of $\mathcal{G}$. There is a unique Lie groupoid structure $[\mathcal{G}:\mathcal{H}] \rightrightarrows M$ such that the blow-down map restricted to $[\mathcal{G}:\mathcal{H}]$ is a base-preserving Lie groupoid morphism

$$p : [\mathcal{G}:\mathcal{H}] \to \mathcal{G}.$$

The exceptional locus $\widetilde{\mathcal{H}} = [\mathcal{G}:\mathcal{H}] \cap p^{-1}(\mathcal{H})$ is then a Lie subgroupoid of codimension 1 in $[\mathcal{G}:\mathcal{H}]$, along which p restricts to a morphism $\widetilde{\mathcal{H}} \to \mathcal{H}$ of Lie groupoids over L.

Proof. To obtain the result, we lift the groupoid structure on $\mathcal{G}$ to maps on $\widetilde{\mathcal{G}} = [\mathcal{G}:\mathcal{H}]$ and verify that the groupoid axioms are satisfied. First, if the blow-down $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ is to be a base-preserving Lie groupoid morphism, the source and target maps of $\widetilde{\mathcal{G}}$ must be given by

$$\widetilde{s} = s \circ p, \qquad \widetilde{t} = t \circ p.$$

In the following steps, we show that $\widetilde{s}, \widetilde{t}$ are submersions, and then obtain the remaining lifts: the multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$ lifts uniquely to $\widetilde{m} : \widetilde{\mathcal{G}} \times_M \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$; the identity $\mathsf{id} : \mathcal{G} \to \mathcal{G}$ lifts uniquely to $\widetilde{\mathsf{id}} : M \to \widetilde{\mathcal{G}}$; and the inverse map $i : \mathcal{G} \to \mathcal{G}$ lifts uniquely to $\widetilde{i} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$.

Once the lifts are defined, we see that they satisfy the groupoid conditions on an open dense set (the complement of $\widetilde{\mathcal{H}}$); by continuity, the groupoid axioms hold on all of $\widetilde{\mathcal{G}}$ and $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ is a Lie groupoid morphism, completing the proof.

Step 1: *The maps $\widetilde{s}, \widetilde{t}$ are submersions of pairs $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{H}}) \to (M, L)$.*

The blow-down $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ is a local diffeomorphism away from $\widetilde{\mathcal{H}}$, so it suffices to show that $T\widetilde{s} : (T\widetilde{\mathcal{G}}|_{\widetilde{\mathcal{H}}}, T\widetilde{\mathcal{H}}) \to (TM|_L, TL)$ is pairwise surjective. Since $Tp : T\widetilde{\mathcal{H}} \to T\mathcal{H}$ and $Ts : T\mathcal{H} \to TL$ are surjective, we immediately obtain that $T\widetilde{s}|_{\widetilde{\mathcal{H}}} : T\widetilde{\mathcal{H}} \to TL$ is surjective.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T\widetilde{\mathcal{H}} & \longrightarrow & T\widetilde{\mathcal{G}}|_{\mathcal{H}} & \longrightarrow & N\widetilde{\mathcal{H}} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle T\widetilde{s}|_{\widetilde{\mathcal{H}}}} & & \Big\downarrow{\scriptstyle T\widetilde{s}} & & \Big\downarrow{\scriptstyle N\widetilde{s}} & & \\ 0 & \longrightarrow & TL & \longrightarrow & TM|_L & \longrightarrow & NL & \longrightarrow & 0 \end{array} \tag{3.1.3}$$

By the commutative diagram (3.1.3), it remains to show that the normal map $N\widetilde{s} : N\widetilde{\mathcal{H}} \to NL$ induced by $T\widetilde{s}$ is surjective. Now, since $s : (\mathcal{G}, \mathcal{H}) \to (M, L)$ is a submersion, the induced map $Ns : N\mathcal{H} \to NL$ is surjective, with kernel subbundle $K_s \subset N\mathcal{H}$, and the composition $N\widetilde{s} = Ns \circ Np$ therefore fails to be surjective along the subset $\mathbb{P}(K_s) \subset \mathbb{P}(N\mathcal{H}) = p^{-1}(\mathcal{H})$ of the exceptional divisor. But $\mathbb{P}(K_s) = p^{-1}(\mathcal{H}) \cap \overline{\overline{s^{-1}L}}$ has been removed in the definition (3.1.2), so that $N\widetilde{s}$ is surjective along $\widetilde{\mathcal{H}}$. The same argument applies to the target map t, yielding the result.

Step 2: *Lifting the multiplication map.*

To lift the multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$ to a map $\widetilde{m} : \widetilde{\mathcal{G}}^{(2)} = \widetilde{\mathcal{G}} \times_M \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$, we apply the universal property of blow-up to the map $f = m \circ (p \times p)$ in order to complete the following commutative diagram.

$$\begin{array}{ccc} \widetilde{\mathcal{G}}^{(2)} & \overset{\widetilde{m}}{\dashrightarrow} & \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}) \\ {\scriptstyle p\times p}\Big\downarrow & \overset{f}{\searrow} & \Big\downarrow{\scriptstyle p} \\ \mathcal{G}^{(2)} & \underset{m}{\longrightarrow} & \mathcal{G} \end{array}$$

By Proposition 2.3.2, it suffices to show that $f^{-1}(\mathcal{H})$ is a hypersurface in $\widetilde{\mathcal{G}}^{(2)}$ whose normal bundle injects into $N\mathcal{H}$ via Nf. First, note that $f^{-1}(\mathcal{H}) = \widetilde{\mathcal{H}}^{(2)} := \widetilde{\mathcal{H}} \times_L \widetilde{\mathcal{H}}$, which is a hypersurface (smooth and codimension 1) by transversality. Then, to show that $Nf : N\widetilde{\mathcal{H}}^{(2)} \to N\mathcal{H}$ is injective, we show the stronger result that $s \circ f$ induces an isomorphism $N\widetilde{\mathcal{H}}^{(2)} \to NL$. Observe that

$$s \circ f = s \circ m \circ (p \times p) = s \circ p_1 \circ (p \times p) = \widetilde{s} \circ \widetilde{p_1},$$

where p_1 and $\widetilde{p_1}$ are the first projections $\mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$ and $\widetilde{\mathcal{G}} \times_M \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$, respectively. The induced map on normal bundles is then the following composition

$$N(\widetilde{\mathcal{H}} \times_L \widetilde{\mathcal{H}}) = N\widetilde{\mathcal{H}} \times_{NL} N\widetilde{\mathcal{H}} \xrightarrow{N\widetilde{p_1}} N\widetilde{\mathcal{H}} \xrightarrow{N\widetilde{s}} NL\ ,$$

where we have used Lemma 3.1.1 to compute $N\widetilde{\mathcal{H}}^{(2)}$. The composition is an isomorphism since $N\widetilde{s}$ is surjective by Step 1 and $N\widetilde{p_1}$ is surjective between bundles of rank 1.

Since $N(s \circ f)$ is an isomorphism, it follows that $Nf(N\widetilde{\mathcal{H}}^{(2)}) \cap K_s = 0$ and $\widetilde{m}(\widetilde{\mathcal{G}}^{(2)}) \cap \mathbb{P}(K_s) = \varnothing$. Similarly $\widetilde{m}(\widetilde{\mathcal{G}}^{(2)}) \cap P(K_t) = \varnothing$ and we obtain $\widetilde{m}(\widetilde{\mathcal{G}}^{(2)}) \subset \widetilde{\mathcal{G}}$.

Step 3: *Lifting the identity and inverse maps.*

We apply Proposition 2.3.2 to the identity map $\mathsf{id} : M \to \mathcal{G}$ and the composition of blow-down and inverse maps $i \circ p : \widetilde{\mathcal{G}} \to \mathcal{G}$ to obtain the lifted identity map $\widetilde{\mathsf{id}} : M \to \mathsf{Bl}_{\mathcal{H}}(\mathcal{G})$ and the lifted inverse map $\widetilde{i} : \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}) \to \mathsf{Bl}_{\mathcal{H}}(\mathcal{G})$. As in Step 2, the image of $\widetilde{\mathsf{id}}$ lies in $\widetilde{\mathcal{G}}$ because $\widetilde{\mathsf{id}}(M) \cap \mathbb{P}(K_s) = \varnothing = \widetilde{\mathsf{id}}(M) \cap \mathbb{P}(K_t)$. The image of $\widetilde{i} : \widetilde{\mathcal{G}} \to \mathsf{Bl}_{\mathcal{H}}(\mathcal{G})$ lies in $\widetilde{\mathcal{G}}$ because $\widetilde{i}$ exchanges $\overline{\overline{s^{-1}(L)}}$ and $\overline{\overline{t^{-1}(L)}}$. ☐

Remark 3.1.3. Using the same idea as **Step 2** in the proof of Theorem 3.1.2, we may extend the multiplication map $m : \widetilde{\mathcal{G}} \times_M \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$ in two ways

$$m_L : \widetilde{\mathcal{G}} \times_M \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}) \to \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}), \qquad m_R : \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}) \times_M \widetilde{\mathcal{G}} \to \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}). \tag{3.1.4}$$

That is, if $\mathcal{G}$ is compact, then $\mathsf{Bl}_{\mathcal{H}}(\mathcal{G})$ is a compactification of $\widetilde{\mathcal{G}}$ such that the left and right action of $\widetilde{\mathcal{G}}$ on itself extends.

3.1.2 Lower elementary modification of Lie algebroids

The blow-up operation for Lie groupoids given in Theorem 3.1.2 corresponds to an operation on Lie algebroids, which we now describe.

Definition 3.1.4. Let L be a closed hypersurface of M. Let $A \to M$ be a vector bundle, and $B \to L$ a subbundle of $A|_L$. We define the ***lower elementary modification*** $[A\!:\!B]$ ***of*** A ***along*** B to be the vector bundle with sheaf of sections given by

$$[A\!:\!B](U) = \{X \in \Gamma(U, A) \mid X|_L \in \Gamma(U \cap L, B)\}, \tag{3.1.5}$$

for open sets $U \subset M$.

The lower elementary modification $[A:B]$ is sometimes called lower Hecke modification, e.g. [44], and $[A\!:\!B]$ is locally free because the ideal sheaf of L, $\mathcal{I}_L$, is locally free of rank 1. For a Lie algebroid (A, M, a) and a Lie subalgebroid (B, L, b) where L is a closed hypersurface of M, the bundle inclusion $B \hookrightarrow A|_L$ is Lie algebroid morphism and Definition 2.1.20 implies that the elementary modification $[A\!:\!B]$ is a Lie algebroid.

Corollary 3.1.5. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and $\mathcal{H} \rightrightarrows L$ a closed Lie subgroupoid over the closed hypersurface L, so that $\mathtt{Lie}(\mathcal{H})$ is a Lie subalgebroid of $\mathtt{Lie}(\mathcal{G})$. Then $\mathtt{Lie}([\mathcal{G}\!:\!\mathcal{H}])$ has sheaf of sections defined by*

$$\mathtt{Lie}([\mathcal{G}\!:\!\mathcal{H}]) = \{X \in \mathtt{Lie}(\mathcal{G}) \mid X|_L \in \mathtt{Lie}(\mathcal{H})\}. \tag{3.1.6}$$

Proof. For any Lie groupoid, we may view the sections of its Lie algebroid as left-invariant vector fields (always taken to be tangent to the source fibers). Therefore, it suffices to show that the blow-down map $p : [\mathcal{G}\!:\!\mathcal{H}] \to \mathcal{G}$ induces a bijection between the right hand side of (3.1.6), viewed as left invariant vector fields on $\mathcal{G}$ tangent to $\mathcal{H}$, and the left hand side of (3.1.6), viewed as left invariant vector fields on $[\mathcal{G}\!:\!\mathcal{H}]$.

Let X be a left-invariant vector field on $\mathcal{G}$ tangent to $\mathcal{H}$, and let $\phi : I \times \mathcal{G} \to \mathcal{G}$ be its flow, defined on a sufficiently small neighbourhood $I \subset \mathbb{R}$ of zero. We show that ϕ lifts to a flow on $[\mathcal{G}\!:\!\mathcal{H}]$ by first lifting the map to the blow-up $\mathsf{Bl}_{I\times\mathcal{H}}(I \times \mathcal{G})$, which completes the commutative diagram (here p, p' are the blow-down maps)

$$\begin{array}{ccc} \mathsf{Bl}_{I\times\mathcal{H}}(I \times \mathcal{G}) & \overset{\widetilde{\phi}}{\dashrightarrow} & \mathsf{Bl}_{\mathcal{H}}(\mathcal{G}) \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ I \times \mathcal{G} & \underset{\phi}{\longrightarrow} & \mathcal{G} \end{array} \tag{3.1.7}$$

and then noting that $\mathsf{Bl}_{I\times\mathcal{H}}(I\times\mathcal{G}) = I\times\mathsf{Bl}_{\mathcal{H}}(\mathcal{G})$, so that $\widetilde{\phi}$ is indeed a flow on $\mathsf{Bl}_{\mathcal{H}}(\mathcal{G})$. Then $\widetilde{X} = \frac{d\widetilde{\phi}}{dt}|_{t=0}$ is the required lift of X to a left invariant vector field on $[\mathcal{G}\!:\!\mathcal{H}]$. The lift $\widetilde{\phi}$ is obtained via Proposition 2.3.2, as follows: since X is tangent to $\mathcal{H}$, we have $\phi^{-1}(\mathcal{H}) = I \times \mathcal{H}$, and so $(\phi \circ p')^{-1}(\mathcal{H})$ is the exceptional divisor in $\mathsf{Bl}_{I\times\mathcal{H}}(I \times \mathcal{G})$, a hypersurface. Furthermore, $N(\phi \circ p')$ is the composition of the injective map Np' and the isomorphism $N\phi$, so is itself injective, proving existence and uniqueness of $\widetilde{\phi}$.

Conversely, we show that $\mathtt{Lie}([\mathcal{G}\,{:}\,\mathcal{H}])$ is generated by the lifts of left-invariant vector fields obtained above. For a sufficiently small neighbourhood $U \subset M$ of $p \in L$, choose a basis of sections of $\mathtt{Lie}(\mathcal{H})$ over $U \cap L$ and extend them to linearly independent sections $(X_1, \ldots, X_l)$ of $\mathtt{Lie}(\mathcal{G})$ over U. Extend this to a basis $(X_1, \ldots, X_l, X_{l+1}, \ldots, X_n)$ of $\mathtt{Lie}(\mathcal{G})$ over U, which we also regard as left invariant vector fields on $\mathcal{G}$. Then, if $f \in C^\infty(U, \mathbb{R})$ is a generator for the ideal sheaf of L in U, we see that

$$(X_1, \ldots, X_l, \widetilde{f}X_{l+1}, \ldots, \widetilde{f}X_n), \tag{3.1.8}$$

for $\widetilde{f} = s^*f$ the pullback of f by the source map of $\mathcal{G}$, forms a C^∞–basis for the right hand side of (3.1.6), showing, incidentally, that it defines a locally free sheaf.

Along the exceptional divisor $E = \mathbb{P}(N\mathcal{H})$ of the blowup $\mathrm{Bl}_{\mathcal{H}}(\mathcal{G})$, the lifts of the vector fields (3.1.8) have determinant given by

$$(p^*d\widetilde{f}|_{\mathcal{H}})^{n-l} \otimes X_1 \wedge \cdots \wedge X_n,$$

where $p^*d\widetilde{f}|_{\mathcal{H}}$ is a function on E. This defines a section of $\det T(\mathrm{Bl}_{\mathcal{H}}(\mathcal{G}))|_E$ which vanishes to order $n-l$ along the bundle of hyperplanes $\{p^*d\widetilde{f}|_{\mathcal{H}} = 0\} \subset E$. But this is precisely the intersection $E \cap \overline{\overline{s^{-1}(L)}}$, which is removed in $[\mathcal{G} : \mathcal{H}] = \mathrm{Bl}_{\mathcal{H}}(\mathcal{G}) \setminus (\overline{\overline{s^{-1}(L)}} \cup \overline{\overline{t^{-1}(L)}})$. Hence the lifts of the vector fields (3.1.8) generate $\mathtt{Lie}([\mathcal{G}\,{:}\,\mathcal{H}])$, as required.

□

In view of Definition 3.1.4, Corollary 3.1.5 may be rephrased to state that there is a canonical isomorphism

$$\mathtt{Lie}([\mathcal{G}\,{:}\,\mathcal{H}]) \cong [\mathtt{Lie}(\mathcal{G})\,{:}\,\mathtt{Lie}(\mathcal{H})]$$

of Lie algebroids, whenever $\mathcal{H} \subset \mathcal{G}$ is a Lie subgroupoid over a closed hypersurface.

Example 3.1.6. Let $\mathcal{G}_0 \rightrightarrows \mathbb{R}^2$ be the pair groupoid of $\mathbb{R}^2$, and choose coordinates (x, y). Let $L = \{x = 0\}$. Then we obtain a subgroupoid $\mathcal{H}_0 = L \times L \subset \mathcal{G}_0$. The Lie algebroid associated to the blow-up $\mathcal{G}_1 = [\mathcal{G}_0\,{:}\,\mathcal{H}_0]$ is

$$\mathtt{Lie}(\mathcal{G}_1) = \left\langle x\tfrac{\partial}{\partial x},\ \tfrac{\partial}{\partial y} \right\rangle,$$

which we recognize as the log tangent algebroid $T(\mathbb{R}^2, L) = [T(\mathbb{R}^2)\,{:}\,TL]$.

Suppose we now blow up $\mathcal{G}_1$ along the codimension 2 subgroupoid $\mathcal{H}_1 = p_1^{-1}(\mathsf{id}_0(L))$, where $p_1 : \mathcal{G}_1 \to \mathcal{G}_0$ is the blow-down map. We have $\mathtt{Lie}(\mathcal{H}_1) = \langle x\frac{\partial}{\partial x}\rangle$, and the Lie algebroid corresponding to $\mathcal{G}_2 = [\mathcal{G}_1\,{:}\,\mathcal{H}_1]$ is then

$$\mathtt{Lie}(\mathcal{G}_2) = \left\langle x\tfrac{\partial}{\partial x},\ x\tfrac{\partial}{\partial y} \right\rangle,$$

which is nothing but the Poisson algebroid $T^*_\pi\mathbb{R}^2$, for $\pi = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$.

On the other hand, we also have a codimension 3 subgroupoid $\mathcal{H}_1' = \mathsf{id}_1(L)$ of $\mathcal{G}_1$. We have $\mathtt{Lie}(\mathcal{H}_1')$ is the trivial Lie algebroid, and the Lie algebroid corresponding to $\mathcal{G}_2' = [\mathcal{G}_1\,{:}\,\mathcal{H}_1']$ is then

$$\mathtt{Lie}(\mathcal{G}_2') = \left\langle x^2\tfrac{\partial}{\partial x},\ x\tfrac{\partial}{\partial y} \right\rangle.$$

The blow-up Lie groupoid satisfies a kind of universal property as described below.

Theorem 3.1.7. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and $\mathcal{H} \rightrightarrows L$ a closed Lie subgroupoid over the closed hypersurface $L \subset M$. Let $\mathcal{F} \rightrightarrows M$ be a source-connected Lie groupoid and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism covering id_M.*

If $\mathtt{Lie}(\varphi)$ factors through the elementary modification $[\mathtt{Lie}(\mathcal{G}) : \mathtt{Lie}(\mathcal{H})] \to \mathtt{Lie}(\mathcal{G})$, then there exists a unique groupoid morphism $\widetilde{\varphi} : \mathcal{F} \to [\mathcal{G} : \mathcal{H}]$ completing the following commutative diagram.

$$\begin{array}{ccc} & & [\mathcal{G}:\mathcal{H}] \\ & \overset{\widetilde{\varphi}}{\nearrow} & \downarrow p \\ \mathcal{F} & \xrightarrow[\varphi]{} & \mathcal{G} \end{array} \tag{3.1.9}$$

Proof. We show that φ satisfies the criterion in Proposition 2.3.2, so lifts to a map $\widetilde{\varphi} : \mathcal{F} \to \mathrm{Bl}_{\mathcal{H}}(\mathcal{G})$. We then show that the image lies in $[\mathcal{G} : \mathcal{H}]$; the remainder follows by continuity.

Let $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ be the Lie algebroids of $\mathcal{F}, \mathcal{G}, \mathcal{H}$, respectively. Observe that L is a $[\mathfrak{g} : \mathfrak{h}]$–invariant submanifold, and since we have a morphism $\mathfrak{f} \to [\mathfrak{g} : \mathfrak{h}]$, it is also a $\mathfrak{f}$–invariant submanifold. Since L is closed, this implies that it is a union of $\mathfrak{f}$–orbits. Since $\mathcal{F}$ is source-connected, it follows that $\mathcal{F}|_L$ is a source-connected Lie subgroupoid of $\mathcal{F}$ of codimension 1. In fact, $\mathcal{F}|_L = \varphi^{-1}(\mathcal{H})$, which can be seen as follows. Since $\mathtt{Lie}(\varphi)$ factors through $\mathtt{Lie}(p) : [\mathfrak{g} : \mathfrak{h}] \to \mathfrak{g}$, we have $\mathtt{Lie}(\varphi)(\mathfrak{f}|_L) \subset \mathfrak{h}$. Therefore, the exponential map gives $\varphi(\mathcal{F}|_L) \subset \mathcal{H}$, since $\mathcal{F}|_L$ is source-connected. For the reverse inclusion, note that if $k \in \mathcal{F} \setminus \mathcal{F}|_L$, then $s(k) \notin L$, so $\varphi(k) \notin \mathcal{H}$.

Since we have shown that $\varphi^{-1}(\mathcal{H})$ is a hypersurface, it remains to show that $N\varphi|_{\mathcal{F}|_L}$ is injective. But the statement follows from the fact that φ intertwines the source maps of $\mathcal{F}$ and $\mathcal{G}$, which are submersions.

Finally, as in Step 2 of Theorem 3.1.2, the image of the lift $\widetilde{\varphi} : \mathcal{F} \to \mathrm{Bl}_{\mathcal{H}}(\mathcal{G})$ lies in $[\mathcal{G} : \mathcal{H}]$, because $N\varphi : N(\varphi^{-1}(\mathcal{H})) \to N\mathcal{H}$ satisfies $\mathsf{image}(N\varphi) \cap K_s = \mathsf{image}(N\varphi) \cap K_t = 0$, where $K_s = \ker(Ns : N\mathcal{H} \to NL)$ and $K_t = \ker(Nt : N\mathcal{H} \to NL)$.

□

Remark 3.1.8. If $\mathcal{G} \rightrightarrows M$ is the adjoint integration of a Lie algebroid $\mathfrak{g}$, and $\mathcal{H} \rightrightarrows L$ is a subgroupoid over a closed hypersurface L, then $[\mathcal{G} : \mathcal{H}]^c$ is the adjoint integration of $[\mathfrak{g} : \mathfrak{h}]$, where $\mathfrak{h} = \mathtt{Lie}(\mathcal{H})$.

In a certain situation, a two-step iterated blow-up of Lie groupoid is a single blow-up, as explained below.

Theorem 3.1.9. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and $\mathcal{K} \subset \mathcal{H}$ an inclusion of subgroupoids, each over the closed hypersurface $L \subset M$. Then the proper transform $\overline{\overline{\mathcal{K}}}$ is a Lie subgroupoid of $[\mathcal{G} : \mathcal{H}]$ over L, and we have a natural isomorphism of groupoids*

$$[\mathcal{G} : \mathcal{K}]^c \cong [[\mathcal{G} : \mathcal{H}] : \overline{\overline{\mathcal{K}}}]^c.$$

Proof. We write the blow-down maps as $p : [\mathcal{G} : \mathcal{H}] \to \mathcal{G}$, $q : [[\mathcal{G} : \mathcal{H}] : \overline{\overline{\mathcal{K}}}] \to [\mathcal{G} : \mathcal{H}]$ and $r : [\mathcal{G} : \mathcal{K}] \to \mathcal{G}$, all of which are Lie groupoid morphisms.

Since $\mathcal{K} \subset \mathcal{H}$, we have $\overline{\overline{\mathcal{K}}} = p^{-1}(\mathcal{K})$. Restrict the source $\widetilde{s} = s \circ p : [\mathcal{G} : \mathcal{H}] \to M$ to $\overline{\overline{\mathcal{K}}}$. Since $p : \overline{\overline{\mathcal{K}}} \to \mathcal{K}$ and $s : \mathcal{K} \to L$ are submersions, it follows that $\widetilde{s} : \overline{\overline{\mathcal{K}}} \to L$ is submersion. Similarly, $\widetilde{t} : \overline{\overline{\mathcal{K}}} \to L$ is a submersion. Since p is a groupoid morphism and $p(\overline{\overline{\mathcal{K}}}) = \mathcal{K}$, it follows that $\widetilde{\mathsf{id}}(L)$ and $\widetilde{m}(\overline{\overline{\mathcal{K}}} \times_L \overline{\overline{\mathcal{K}}})$ both lie in $\overline{\overline{\mathcal{K}}}$. This shows $\overline{\overline{\mathcal{K}}}$ is a Lie subgroupoid of $[\mathcal{G} : \mathcal{H}]$.

Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ be the Lie algebroids of $\mathcal{G}$, $\mathcal{H}$, $\mathcal{K}$, respectively. Since $\mathfrak{k} \subset \mathfrak{h}$ and $\mathfrak{h}$ is a Lie subalgebroid of $\mathfrak{g}$, it follows from Definition 3.1.4 that

$$[\mathfrak{g} : \mathfrak{k}] = [[\mathfrak{g} : \mathfrak{h}] : \mathfrak{k}].$$

Since $\mathtt{Lie}(p \circ q) : [[\mathfrak{g} : \mathfrak{h}] : \mathfrak{k}] \to \mathfrak{g}$ factors through $[\mathfrak{g} : \mathfrak{k}]$, by Theorem 3.1.7, we obtain a Lie groupoid morphism

$$\varphi : [[\mathcal{G} : \mathcal{H}] : \overline{\overline{\mathcal{K}}}]^c \to [\mathcal{G} : \mathcal{K}].$$

Likewise, $\mathtt{Lie}(r) : [\mathfrak{g} : \mathfrak{k}] \to \mathfrak{g}$ factors through $[[\mathfrak{g} : \mathfrak{h}] : \mathfrak{k}]$, so we obtain the morphism

$$\phi : [\mathcal{G} : \mathcal{K}]^c \to [[\mathcal{G} : \mathcal{H}] : \overline{\overline{\mathcal{K}}}].$$

Since $\varphi \circ \phi : [\mathcal{G} : \mathcal{K}]^c \to [\mathcal{G} : \mathcal{K}]^c$ is a Lie groupoid morphism covering id on $[\mathfrak{g} : \mathfrak{k}]$, it follows that $\varphi \circ \phi$ is an automorphism of $[\mathcal{G} : \mathcal{K}]^c$, showing that $[\mathcal{G} : \mathcal{K}]^c \cong [[\mathcal{G} : \mathcal{H}] : \overline{\overline{\mathcal{K}}}]^c$. □

3.2 Blow-up of Poisson groupoids

In this section, we specializes the results of §3.1 to Poisson groupoids.

3.2.1 Blow-up of Poisson manifolds

Let (M, π) be a Poisson manifold, and let $L \subset M$ be a Poisson submanifold. The normal space N_pL to any point $p \in L$ then inherits a linear Poisson structure, defining a transverse Poisson structure

$$\pi_N \in \Gamma(L, N^*L \otimes \wedge^2 NL), \tag{3.2.1}$$

which exhibits the conormal bundle N^*L as a bundle of Lie algebras.

In [32], Polishchuk observed that in order for the Poisson structure on M to lift to the blow-up $\mathsf{Bl}_L(M)$, the transverse Poisson structure along L must be *degenerate*, in the following sense.

Definition 3.2.1. The transverse Poisson structure π_N of a Poisson submanifold $(L, \pi|_L) \subset (M, \pi)$ is called *degenerate* when

$$\pi_N = v \wedge E,$$

where v is the normal vector field obtained from π_N via the contraction $N^*L \otimes \wedge^2 NL \to NL$, and E is the Euler vector field on NL.

Remark 3.2.2. The Lie algebra on each conormal space N_p^*L induced by a degenerate transverse Poisson structure is either abelian, when $v(p) = 0$, or isomorphic to the semidirect product Lie algebra $\mathbb{R} \ltimes \mathbb{R}^{n-1}$ associated to the action $a \cdot u = au$ of $\mathbb{R}$ on $\mathbb{R}^{n-1}$, where $n = \operatorname{codim} L$.

Theorem 3.2.3 (Polishchuk [32])**.** *Let $(L, \pi|_L) \subset (M, \pi)$ be a closed Poisson submanifold with degenerate transverse Poisson structure π_N. Then there is a unique Poisson structure $\widetilde{\pi}$ on $\mathsf{Bl}_L(M)$ such that $p_*(\widetilde{\pi}) = \pi$. Furthermore, the exceptional divisor is Poisson if and only if π_N vanishes.*

3.2.2 Lifting theorem

We apply Theorem 3.2.3 to Poisson groupoids.

Theorem 3.2.4. *Let $(\mathcal{G} \rightrightarrows M, \sigma)$ be a Poisson groupoid such that*

$$\texttt{Lie}(\mathcal{G}, \sigma) = (A, A^*). \tag{3.2.2}$$

Let $L \subset M$ be a closed hypersurface, and let $\mathcal{H} \rightrightarrows L$ be a Poisson subgroupoid such that

$$\texttt{Lie}(\mathcal{H}, \sigma_{\mathcal{H}}) = (B, B^*). \tag{3.2.3}$$

If the induced transverse Poisson structure on $N^*\mathcal{H}$ is degenerate, then the blow-up Lie groupoid $[\mathcal{G} : \mathcal{H}] \rightrightarrows M$ inherits a multiplicative Poisson structure σ' such that

$$\texttt{Lie}([\mathcal{G}:\mathcal{H}], \sigma') = ([A:B], [A:B]^*). \tag{3.2.4}$$

where $[A:B]$ is the lower modification of A along B and $[A:B]^*$ is the dual vector bundle.

The Lie algebroid $[A : B]^*$ can be described as an upper elementary modification, which we now describe.

Definition 3.2.5. Let $A \to M$ be a vector bundle, and let $B \to L$ be a vector bundle over a closed hypersurface $L \subset M$ with a surjective bundle morphism

$$\phi : A|_L \to B. \tag{3.2.5}$$

Let $K = \ker(\phi)$ be the kernel. Let f be a defining function of L, i.e. $f|_L = 0$ and $df|_L \neq 0$.

We define the ***upper elementary modification*** $\{A:B\}$ of A along B to be the vector bundle with sheaf of sections given by

$$\Gamma(\{A:B\}) = \{X \in \Gamma(A) \otimes \mathcal{I}_L^{-1} \mid fX|_L \in \Gamma(K)\}. \tag{3.2.6}$$

Remark 3.2.6. Definition 3.2.5 does not depend on the choice of the defining function f.

Recall that that in the case of a lower elementary modification of Lie algebroids $[A:B]$, we require that B is a Lie subalgebroid of A over a closed hypersurface L. In other words, we require that B is a subbundle of $A|_L$, and the bundle inclusion $\iota : B \hookrightarrow A$ is a Lie algebroid morphism. For an upper elementary modification of Lie algebroids $\{A : B\}$ to be a Lie algebroid, we need the dual condition, namely, $\phi : A|_L \to B$ is a surjective Lie algebroid comorphism.

Definition 3.2.7. Let $B \to L$ and $A \to M$ be Lie algebroids, and let $\iota : L \to M$ be a smooth map. A ***comorphism of Lie algebroids*** over ι is a vector bundle morphism $\phi : \iota^* A \to B$ satisfying

- $\iota^*(a(X)(u)) = b(\phi(X))(\iota^*(u))$ for $u \in C^\infty(M)$ and $X \in \Gamma(A)$; $(\star)$
- $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for $X, Y \in \Gamma(A)$. $(\star\star)$

Proposition 3.2.8. *Let $A \to M$ be a Lie algebroid, and let $B \to L$ be a Lie algebroid over a closed hypersurface L with a surjective comorphism $\phi : A|_L \to B$. Then the upper modification $\{A:B\}$ has a Lie algebroid structure such that the natural bundle map $A \to \{A:B\}$ is a Lie algebroid morphism.*

Proof. Condition ($\star$) implies that $K \subset \ker(a|_L : A|_L \to TM|_L)$. This means for $X \in \Gamma(A)$ such that $X|_L \in \Gamma(K)$, we have $a(X) \in \Gamma(TM(-L))$. Therefore, the anchor $a : A \to TM$ lifts to $a' : \{A\!:\!B\} \to TM$. Condition ($\star\star$) implies that the bracket on $\Gamma(A)$ lifts to a bracket on $\Gamma(\{A\!:\!B\})$. □

Example 3.2.9. Consider $M = \mathbb{R}^2$ and the y-axis $L = \{x{=}0\}$. Let $A = TM$ and $B = TL$ with the surjective bundle morphism

$$\phi : A|_L \to B, \qquad \tfrac{\partial}{\partial x} \mapsto 0, \quad \tfrac{\partial}{\partial y} \mapsto \tfrac{\partial}{\partial y}. \tag{3.2.7}$$

Then the upper modification of vector bundles $\{A\!:\!B\}$ exists and

$$\{A\!:\!B\} =< \tfrac{1}{x}\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} > . \tag{3.2.8}$$

However, ϕ fails to be a comorphism of Lie algebroids. Neither ($\star$) and ($\star\star$) are satisfied. The anchor image of $\frac{1}{x}\frac{\partial}{\partial x}$ is not well-defined, so there is no anchor map. Also, we have

$$\left[\tfrac{1}{x}\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}\right] = \tfrac{1}{x^2}\tfrac{\partial}{\partial x} \notin \Gamma(\{A\!:\!B\}), \tag{3.2.9}$$

so the bracket is not closed. In conclusion, $\{A\!:\!B\}$ is not a Lie algebroid.

Example 3.2.10. Consider $M = \mathbb{R}^2$ and the y-axis $L = \{x{=}0\}$. Let $A = [TM\!:\!TL]$ and $B = TL$ with the surjective bundle morphism

$$\phi : A|_L \to B, \qquad x\tfrac{\partial}{\partial x} \mapsto 0, \quad \tfrac{\partial}{\partial y} \mapsto \tfrac{\partial}{\partial y}. \tag{3.2.10}$$

Then ϕ is a comorphism of Lie algebroids and the upper modification

$$\{A\!:\!B\} =< \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} > \tag{3.2.11}$$

is the tangent algebroid TM.

Now since $[A\!:\!B]^* = \{A^*\!:\!B^*\}$, we may rephrase Theorem 3.2.4 in the following way.

Theorem 3.2.11. *Let $(\mathcal{G} \rightrightarrows M, \sigma)$ be a Poisson groupoid such that*

$$\texttt{Lie}(\mathcal{G}, \sigma) = (A, A^*). \tag{3.2.12}$$

Let $L \subset M$ be a closed hypersurface and let $\mathcal{H} \rightrightarrows L$ be a Poisson subgroupoid such that

$$\texttt{Lie}(\mathcal{H}, \sigma_{\mathcal{H}}) = (B, B^*). \tag{3.2.13}$$

If the induced transverse Poisson structure on $N^*\mathcal{H}$ is degenerate, then the induced bundle map

$$A^*|_L \longrightarrow B^*$$

is a surjective Lie algebroid comorphism and the blow-up Lie groupoid $[\mathcal{G}\!:\!\mathcal{H}] \rightrightarrows M$ inherits a multiplicative Poisson structure σ' such that

$$\texttt{Lie}([\mathcal{G}\!:\!\mathcal{H}], \sigma') = ([A\!:\!B], \{A^*\!:\!B^*\}) \tag{3.2.14}$$

where $[A\!:\!B]$ *is the lower modification of* A *along* B *and* $\{A^*\!:\!B^*\}$ *is the upper modification of* A^* *along* B^*.

3.3 Fiber products

In this subsection, we introduce the fiber product construction of Lie groupoids, generalizing the 'puff' construction of Monthubert [30].

Definition 3.3.1. Let M be a smooth manifold, and let $L_1, L_2, \ldots, L_n$ be closed hypersurfaces. We say $L_1, L_2, \ldots, L_n$ are ***normal crossing*** if for $x \in L_{i_1} \cap L_{i_2} \cap \ldots \cap L_{i_m}$, the conormal lines $N_x^* L_{i_1}, N_x^* L_{i_2}, \ldots, N_x^* L_{i_m}$ are linearly independent in $T_x^* M$.

Following the language of algebraic geometry, we write a collection of normal crossing hypersurfaces as $D = L_1 + L_2 + \ldots + L_n$, and call D a ***normal crossing divisor*** of M.

Remark 3.3.2. For a normal crossing divisor $L_1 + L_2 + \ldots + L_n$ of M, the intersection of m hypersurfaces $L_{i_1} \cap L_{i_2} \cap \ldots \cap L_{i_m}$ is either empty or an embedded submanifold of dimension $n - m$.

Proposition 3.3.3. *Let* (A, M, a) *be a Lie algebroid, and let* (B_1, L_1) *and* (B_2, L_2) *be Lie subalgebroids where* $L_1 + L_2$ *is a normal crossing divisor.*

For $L_{12} = L_1 \cap L_2$, *we have that* $B_{12} = B_1|_{L_{12}} \cap B_2|_{L_{12}}$ *is a Lie subalgebroid of* B_2 *over* L_{12}. *Moreover,* $[B_2\!:\!B_{12}]$ *is a Lie subalgebroid of* $[A\!:\!B_1]$ *over* L_2.

Sketch of the proof. The fact that B_{12} is a Lie subalgebroid of B_2 is a straight-forward check of Definition 2.1.20. The fact that $[B_2\!:\!B_{12}]$ is a Lie subalgebroid of $[A\!:\!B_1]$ over L_2 follows from Remark 2.1.21. □

Let (A, M, a) be a Lie algebroid, and let $(B_1, L_1), \ldots, (B_n, L_n)$ be Lie subalgebroids where $L_1 + L_2 + \ldots + L_n$ is a normal crossing divisor. For $L_{ij} = L_i \cap L_j \neq \varnothing$, we write $B_{ij} = B_i|_{L_{ij}} \cap B_j|_{L_{ij}}$.

We define the ***Lie algebroid lower modification*** of A along $B_1, \ldots, B_n$, denoted by $[A\!:\!B_1, \ldots, B_n]$, inductively as follows.

$$
\begin{aligned}
[A\!:\!B_1] &= [A\!:\!B_1] \\
[A\!:\!B_1, B_2] &= [[A\!:\!B_1]\!:\![B_2\!:\!B_{12}]] \\
[A\!:\!B_1, B_2, B_3] &= [[A\!:\!B_1, B_2]\!:\![B_3\!:\!B_{13}, B_{23}]] \\
&\cdots \\
[A\!:\!B_1, B_2, \ldots, B_n] &= [[A\!:\!B_1, B_2, \ldots, B_{n-1}]\!:\![B_n\!:\!B_{1n}, B_{2n}, \ldots, B_{(n-1)n}]]
\end{aligned}
$$

Lemma 3.3.4. *With the above notations, if the bundle maps* $q_1 : [A\!:\!B_1] \to A$ *and* $q_2 : [A\!:\!B_2] \to A$ *are transversal, then as a vector bundle, we have*

$$[A\!:\!B_1, B_2] = [A\!:\!B_1] \oplus_A [A\!:\!B_2].$$

Proof. We fix the local generators for the vector bundles B_{12}, B_1, B_2 and A as follows:

$$\begin{aligned} B_{12} &=< Z_1, \dots Z_k >, \\ B_1 &=< Z_1, \dots, Z_k, X_1, \dots, X_l >, \\ B_2 &=< Z_1, \dots, Z_k, X_{l+1}, \dots, X_m >, \\ A &=< Z_1, \dots, Z_k, X_1, \dots, X_m, W_1, \dots, W_n > . \end{aligned}$$

Now assume locally, the hypersurface L_1 is parametrized by $x = 0$, and L_2 is parametrized by $y = 0$. Then, we have

$$\begin{aligned} [A\!:\!B_1] &=< Z_1, \dots, Z_k, X_1, \dots, X_l, xX_{l+1}, \dots xX_m, xW_1, \dots, xW_n >, \\ [B_2\!:\!B_{12}] &=< Z_1, \dots, Z_k, xX_{l+1}, \dots xX_m >, \\ [[A\!:\!B_1]\!:\![B_2\!:\!B_{12}]] &=< Z_1, \dots, Z_k, yX_1, \dots, yX_l, xX_{l+1}, \dots xX_m, xyW_1, \dots, xyW_n > . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [A\!:\!B_1] &=< Z_1, \dots, Z_k, X_1, \dots, X_l, xX_{l+1}, \dots xX_m, xW_1, \dots, xW_n >, \\ [A\!:\!B_2] &=< Z_1, \dots, Z_k, yX_1, \dots, yX_l, X_{l+1}, \dots X_m, yW_1, \dots, yW_n >, \\ [A\!:\!B_1] \oplus_A [A\!:\!B_2] &=< Z_1, \dots, Z_k, yX_1, \dots, yX_l, xX_{l+1}, \dots xX_m, xyW_1, \dots, xyW_n > . \end{aligned}$$

□

Applying induction, we have the following result.

Proposition 3.3.5. *With the above notations, if the bundle maps $q_j : [A\!:\!B_j] \to A$ are transversal, then as vector bundles, we have*

$$[A\!:\!B_1, \dots, B_n] = [A\!:\!B_1] \oplus_A [A\!:\!B_2] \oplus_A \dots \oplus_A [A\!:\!B_n]. \tag{3.3.1}$$

In particular, $[A\!:\!B_1, \dots, B_n]$ is independent of the ordering of B_i.

Next, we introduce the fiber product of Lie groupoids, c.f. the strong fiber product of Lie groupoids in [29] Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $\mathcal{H}_1 \rightrightarrows L_1$ and $\mathcal{H}_2 \rightrightarrows L_2$ be a Lie subgroupoids where L_1 and L_2 are normal crossing hypersurfaces. Let $\mathcal{G}_1 = [\mathcal{G}\!:\!\mathcal{H}_1]$ and $\mathcal{G}_2 = [\mathcal{G}\!:\!\mathcal{H}_2]$ be the blow-up groupoids as in Theorem 3.1.2 with the blow-down maps $p_1 : \mathcal{G}_1 \to \mathcal{G}$ and $p_2 : \mathcal{G}_2 \to \mathcal{G}$. We define a groupoid structure on the fiber product

$$\widetilde{\mathcal{G}} = \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 = \{(g_1, g_2) \times \mathcal{G}_1 \times \mathcal{G}_2 \mid p_1(g_1) = p_2(g_2)\}$$

over the base manifold M as follows. For $(g_1, g_2), \in \widetilde{\mathcal{G}}$, $((g_1, g_2), (k_1, k_2)) \in \widetilde{\mathcal{G}}_{\widetilde{s}}\times_{\widetilde{t}}\widetilde{\mathcal{G}}$ and $x \in M$, we have

$$\begin{aligned} &\widetilde{s}(g_1, g_2) = s_1(g_1), \quad \widetilde{t}(g_1, g_2) = t_2(g_2), \quad \widetilde{m}\left((g_1, g_2), (k_1, k_2)\right) = (g_1, k_2), \\ &\widetilde{\mathsf{id}}(x) = (\mathsf{id}_1(x), \mathsf{id}_2(x)), \quad (g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1}). \end{aligned}$$

In general, $p_1 : \mathcal{G}_1 \to \mathcal{G}$ and $p_2 : \mathcal{G}_2 \to \mathcal{G}$ are not necessarily transversal, but if they are, then taking the Lie functor, we have that Lie algebroid morphisms $\mathtt{Lie}(p_1) : \mathtt{Lie}(\mathcal{G}_1) \to \mathtt{Lie}(\mathcal{G})$ and $\mathtt{Lie}(p_2) :$

$\mathtt{Lie}(\mathcal{G}_2) \to \mathtt{Lie}(\mathcal{G})$ are also transversal. Writing $A = \mathtt{Lie}(\mathcal{G})$ $A_1 = \mathtt{Lie}(\mathcal{G}_1)$ and $A_2 = \mathtt{Lie}(\mathcal{G}_2)$, the Lie functor commutes with the fiber product, i.e.

$$\mathtt{Lie}(\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2) = A_1 \oplus_A A_2. \tag{3.3.2}$$

Applying induction, we obtain the following result.

Proposition 3.3.6. *With the above notations, let $\mathcal{G} \rightrightarrows M$ is a Lie groupoid integrating A, and for $j = 1, \ldots, n$, let $\mathcal{H}_j$ be the Lie subgroupoid integrating B_j. For each j, we have that $\mathcal{G}_j = [\mathcal{G} : \mathcal{H}_j]$ is the blow-up groupoid as in Theorem 3.1.2. If the blow-down maps $p_j : \mathcal{G}_j \to \mathcal{G}$ is transversal, then we have*

$$\mathtt{Lie}\left([\mathcal{G}:\mathcal{H}_1]\times_{\mathcal{G}} \ldots \times_{\mathcal{G}}[\mathcal{G}:\mathcal{H}_n]\right) = [A:B_1] \oplus_A \ldots \oplus_A [A:B_n].$$

Alternatively, we may obtain a groupoid integrating $[A:B_1, \ldots, B_n]$ by iterated blow up's as follows

1. There is a Lie subgroupoid $\mathcal{K}_1$ of $\mathcal{G}_1$ integrating $[B_2:B_{12}]$, and the blow-up groupoid $[\mathcal{G}:\mathcal{H}_1, \mathcal{H}_2] = [\mathcal{G}_1:\mathcal{K}_1]$ is a Lie groupoid integrating $[A:B_1, B_2]$;

2. There is a Lie subgroupoid $\mathcal{K}_2$ of $\mathcal{G}_2'$ integrating $[B_3 : B_{13}, B_{23}]$, and the blow-up groupoid $[\mathcal{G} : \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3] = [\mathcal{G}_2:\mathcal{K}_2]$ is a Lie groupoid integrating $[A:B_1, B_2]$;

   ...

n. ... and the blow-up groupoid $[\mathcal{G} : \mathcal{H}_1, \ldots, \mathcal{H}_n] = [\mathcal{G}_{n-1} : \mathcal{K}_{n-1}]$ is a Lie groupoid integrating $[A:B_1, \ldots, B_n]$.

To conclude this chapter, we prove the groupoid analogue to Proposition 3.3.5.

Lemma 3.3.7. *With the above notations, we have*

$$[\mathcal{G}:\mathcal{H}_1, \mathcal{H}_2]^c = \left([\mathcal{G}:\mathcal{H}_1]\times_{\mathcal{G}}[\mathcal{G}:\mathcal{H}_2]\right)^c.$$

Proof. Let us write

$$\widetilde{\mathcal{G}} = \left([\mathcal{G}:\mathcal{H}_1]\times_{\mathcal{G}}[\mathcal{G}:\mathcal{H}_2]\right)^2 = \left(\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2\right)^c, \quad \mathcal{G}_2' = [\mathcal{G}:\mathcal{H}_1, \mathcal{H}_2]^c,$$

and let $p_2' : \mathcal{G}_2' \to \mathcal{G}$ be the composition of blow-down groupoid morphism. We use the universal property of fiber products to construct a groupoid morphism $\phi : \mathcal{G}_2' \to \widetilde{\mathcal{G}}$, and then use Theorem 3.1.7 to construct another groupoid morphism $\psi : \widetilde{\mathcal{G}} \to \mathcal{G}_2'$, which must be inverse to ϕ.

Since $\mathtt{Lie}(\mathcal{G}_2') = [A : B_1, B_2]$ and $[A : B_1, B_2] = [[A : B_2] : [B_1 : B_{12}]]$, it follows that $\mathtt{Lie}(p_2')$ factors through $[A:B_2]$. By Theorem 3.1.7, we obtain a Lie groupoid morphism $f_{12} : \mathcal{G}_2' \to \mathcal{G}_2$. That is, we have a commutative diagram as follows

$$\begin{array}{ccc} \mathcal{G}_2' & \longrightarrow & \mathcal{G}_1 \\ {\scriptstyle f_{12}}\downarrow & & \downarrow \\ \mathcal{G}_2 & \longrightarrow & \mathcal{G} \end{array}$$

By the universal property of fiber products, we obtain a groupoid morphism $\phi : \mathcal{G}_2' \to \widetilde{\mathcal{G}}$.

On the other hand, note that $\mathcal{G}_2' = [\mathcal{G}_1 : \mathcal{K}_1]$ is the blow-up groupoid and the natural projection $\widetilde{\mathcal{G}} \to \mathcal{G}_1$ is a groupoid morphism. Since $\mathtt{Lie}(\widetilde{\mathcal{G}}) = \mathtt{Lie}(\mathcal{G}_2')$, by Theorem 3.1.7, we obtain a groupoid morphism $\psi : \widetilde{\mathcal{G}} \to \mathcal{G}_2'$. □

Apply induction, we have the following result.

Proposition 3.3.8. *With the above notations, we have*

$$[\mathcal{G}:\mathcal{H}_1, \ldots, \mathcal{H}_n]^c = \left([\mathcal{G}:\mathcal{H}_1]\times_{\mathcal{G}} \ldots \times_{\mathcal{G}}[\mathcal{G}:\mathcal{H}_n]\right)^c. \tag{3.3.3}$$

Chapter 4

Log symplectic manifolds

Poisson manifolds of log symplectic type were studied by Goto in the holomorphic category [11], and by Guillemin, Miranda and Pires in the smooth category [15, 16]. In dimension 2, Radko provided a complete classification [34]. These Poisson manifolds are generically symplectic, and degenerate along a hypersurface. In §5.2, we describe the Poisson geometry near such a hypersurface. In §4.2, we recall, from [34], the classification of the orientable log symplectic surfaces, and extend it to the non-orientable log symplectic surfaces.

4.1 Poisson geometry of the degeneracy locus

Definition 4.1.1. A *log symplectic manifold* is a smooth $2n$–manifold M, equipped with a Poisson structure π whose Pfaffian, π^n, vanishes transversely.[1]

The degeneracy locus $D = (\pi^n)^{-1}(0)$ is then an embedded, possibly disconnected, Poisson hypersurface, and $M \setminus D$ is a union of open symplectic leaves. If M is compact, then both D and $M \setminus D$ have finitely many components. The Poisson structure π is called log symplectic because π^{-1} defines a logarithmic symplectic form, as we now explain.

Definition 4.1.2. The *log tangent bundle* $T(M, D)$ associated to a closed hypersurface $D \subset M$ is the vector bundle associated to the sheaf of vector fields on M tangent to D.

Equipped with the induced Lie bracket and the inclusion morphism $a : T(M, D) \to TM$, it is a Lie algebroid. In fact, we have $T(M, D) = [TM : TD]$.

Remark 4.1.3. The de Rham complex of the Lie algebroid $T(M, D)$ may be interpreted as differential forms with logarithmic singularities along D; it was introduced in [9] and is denoted by $(\Omega^\bullet_M(\log D), d)$.

Proposition 4.1.4. *Let (M, π) be a log symplectic manifold with degeneracy locus D. Then π^{-1} determines a nondegenerate closed logarithmic 2-form in $\Omega^2_M(\log D)$.*

Proof. If f is a local smooth function vanishing to first order along D, then $\pi(df)$ must vanish along D, since D is Poisson. Therefore, $\pi(df) = fY$ for a smooth vector field Y, which must be tangent to D,

[1] That is, for $x \in M$ such that $\pi^n(x) = 0$, we have $d\pi^n(x) \neq 0$.

since $Y(f) = f^{-1}\pi(df, df) = 0$. This proves that $\pi : T^*M \to TM$ lifts to $\widetilde{\pi} : T^*(M,D) \to T(M,D)$, commuting with the natural inclusions:

$$\begin{array}{ccc} T^*(M,D) & \xrightarrow{\widetilde{\pi}} & T(M,D) \\ \uparrow{\scriptstyle a^*} & & \downarrow{\scriptstyle a} \\ T^*M & \xrightarrow[\pi]{} & TM \end{array}$$

It remains to show $\widetilde{\pi}$ is an isomorphism, but this is obtained from the determinant of the above diagram: $\det a$ and $\det a^*$ vanish to first order along D, whereas $\det \pi = \pi^n \otimes \pi^n$ vanishes to second order. Hence $\widetilde{\pi}$ is nondegenerate, and $\widetilde{\pi}^{-1}$ is closed since π^{-1} is a well-defined symplectic form on $M \backslash D$. □

To describe the geometry of log symplectic manifolds in a neighbourhood of the degeneracy locus, we make use of the notion of a rank 1 Poisson module [32] or Poisson line bundle, which is a line bundle with a flat Lie algebroid connection with respect to the Poisson algebroid $T^*_\pi M$.

Definition 4.1.5. A ***Poisson vector bundle*** over the Poisson manifold (M, π) is a vector bundle $V \to M$ equipped with a flat Poisson connection, i.e. a differential operator $\partial : \Gamma(V) \to \Gamma(TM \otimes V)$ such that $\partial(fs) = \pi(df) \otimes s + f\partial s$ for $f \in C^\infty(M)$ and with vanishing curvature in $\Gamma(\wedge^2 TM)$.

A real line bundle L always admits a flat connection ∇, and any flat Poisson connection ∂ may be written

$$\partial = \pi \circ \nabla + Z, \tag{4.1.1}$$

for Z a Poisson vector field. Another flat connection ∇' differs from ∇ by a closed real 1-form A, so that the Poisson vector field Z is determined uniquely by ∂ only up to the addition of a locally Hamiltonian vector field. For this reason, if the underlying Poisson manifold has odd dimension $2n-1$, the multivector $Z \wedge \pi^{n-1}$ is independent of the choice of ∇. We call this the residue of (L, ∂), following [14].

Definition 4.1.6. The ***residue*** $\chi \in \Gamma(\wedge^{2n-1}TD)$ of a Poisson line bundle (L, ∂) over a Poisson $(2n-1)$–manifold (D, σ) is defined by

$$\chi = Z \wedge \sigma^{n-1},$$

where Z is a Poisson vector field given by (4.1.1).

Example 4.1.7. The anti-canonical bundle $K^* = \det TM$ is a Poisson line bundle, with Poisson connection ∂ uniquely determined by the condition

$$\partial_{df}(\rho) = \mathcal{L}_{\pi(df)}\rho, \tag{4.1.2}$$

where $f \in C^\infty(M)$ and $\rho \in \Gamma(K^*)$. In coordinates where $\pi = \pi^{ij}\partial_{x_i} \wedge \partial_{x_j}$, the associated Poisson vector field via 4.1.1 is $Z = (\partial_k \pi^{ik})\partial_{x_i}$, known as the modular vector field [42] associated to the Lebesgue measure.

We now describe the Poisson geometry of the degeneracy locus D, recovering some results of [15] by slightly different means.

Proposition 4.1.8. *The degeneracy locus D of a log symplectic $2n$-manifold (M, π) is a $(2n-1)$-manifold whose Poisson structure has constant rank $2n-2$ and which admits a Poisson vector field transverse to the symplectic foliation. In particular, D is unimodular.*

Proof. Since π^n vanishes transversely along D, its first derivative (i.e. first jet) $j^1(\pi^n)$ defines, along D, a nonvanishing section χ of $N^*D \otimes \wedge^{2n}TM|_D \cong \wedge^{2n-1}TD$, i.e. a covolume form on D. On the other hand, the Leibniz rule for any connection on TM gives $\nabla\pi^n = n\pi^{n-1} \wedge \nabla\pi$, so that π^{n-1} is nonvanishing along D, showing π has rank $2n-2$ along D.

By Example 4.1.7, the anti-canonical bundle $K^* = \det TM$ has a natural flat Poisson connection ∂. Upon choosing a usual flat connection ∇ on K^*, we obtain, via (4.1.1), a Poisson vector field Z on M, which must be tangent to the degeneracy locus D. It remains to show that Z is transverse to the symplectic leaves on D. This may be rephrased as follows: the anti-canonical bundle restricts to a Poisson line bundle on D, canonically the normal bundle ND, and we claim its residue $Z \wedge \pi^{n-1}$ is nonvanishing. In fact, we show it coincides with the covolume χ defined above.

To see this, choose a flat local trivialization ρ for K^* near a point in D. Then $\pi^n = f\rho$, for a smooth function f. From (4.1.2), we have $\partial\pi^n = 0$, and applying (4.1.1), we obtain $\pi(df) + fZ = 0$, i.e. Z has singular Hamiltonian $-\log f$. Therefore:

$$\chi = \mathrm{Tr}(\nabla\pi^n)|_D = \mathrm{Tr}(\nabla(f\rho))|_D = (i_{d\log f}\pi^n)|_D = n(Z \wedge \pi^{n-1})|_D,$$

showing that $Z \wedge \pi^{n-1}$ is nonvanishing on D, as required. Also, since Z is Poisson, this covolume form is invariant under Hamiltonian flows, i.e. D is unimodular. □

Proposition 4.1.8 has a converse, because the total space of a Poisson line bundle is naturally Poisson [32]. This provides an alternative approach to the extension theorem for regular corank one Poisson structures in [16].

Proposition 4.1.9. *Let (D, σ) be a Poisson $(2n-1)$–manifold of constant rank $2n-2$, and let N be a Poisson line bundle with nonvanishing residue $\chi \in \Gamma(\wedge^{2n-1}TD)$. Then the total space of N is a log symplectic manifold with degeneracy locus (D, σ).*

Proof. Choose a flat connection[2] ∇ on N, and let Z be given by (4.1.1). Let $\widetilde{\sigma}$ and $\widetilde{Z}$ be the horizontal lifts of σ and Z to the total space $\mathrm{tot}(N)$ of N, and let E be the Euler vector field. Then

$$\pi = \widetilde{\sigma} + \widetilde{Z} \wedge E \tag{4.1.3}$$

is a log symplectic structure on $\mathrm{tot}(N)$ with degeneracy locus (D, σ). We now verify that π is independent of the choice of ∇: for another connection $\nabla' = \nabla + A$, the horizontal lifts differ by $\widetilde{X}' - \widetilde{X} = A(X)E$, so that

$$\widetilde{\sigma}' = \widetilde{\sigma} + \sigma(A) \wedge E.$$

On the other hand, for the new connection $Z' = Z - \sigma(A)$, so that

$$\widetilde{Z}' \wedge E = \widetilde{Z} \wedge E - \sigma(A) \wedge E,$$

showing (4.1.3) is independent of ∇.

□

[2]Proposition 4.1.9 also holds for complex line bundles [32], when flat connections may not exist; for later convenience we present an argument tailored to the real case.

In the proof of Proposition 4.1.8, we saw that the normal bundle of the degeneracy locus is itself a Poisson line bundle. By Proposition 4.1.9, therefore, its total space inherits a natural log symplectic structure. This structure is called the ***linearization*** of π along D. We require a very concrete description of this linearized Poisson structure, so we restrict to a special class of log symplectic manifolds.

Definition 4.1.10. A log symplectic manifold is ***proper*** when each connected component D_j of its degeneracy locus $D = \coprod_j D_j$ is compact and contains a compact symplectic leaf F_j.

Now let (M, π) be a proper log symplectic manifold, with connected degeneracy locus D, containing a compact symplectic leaf (F, ω). As shown in [15], it follows from the Reeb–Thurston stability theorem that the transverse Poisson vector field Z renders D isomorphic to a symplectic mapping torus $S^1_\lambda \ltimes_\varphi F$:

$$S^1_\lambda \ltimes_\varphi F = \frac{F \times \mathbb{R}}{(x,t) \sim (\varphi(x), t+\lambda)}, \quad \lambda > 0, \tag{4.1.4}$$

where $\varphi : F \to F$ is a symplectomorphism. Note that there is a natural projection map

$$f : S^1_\lambda \ltimes_\varphi F \longrightarrow S^1_\lambda = \frac{\mathbb{R}}{t \sim t+\lambda},$$

and the Poisson structure on the mapping torus is given by $\iota\omega^{-1}\iota^*$, where ι is the inclusion morphism of the subbundle $\ker(Tf) \subset TD$. The normal bundle of D is a real line bundle, classified up to isomorphism by $H^1(D, \mathbb{Z}_2) = H^1(F, \mathbb{Z}_2)^\varphi \times \mathbb{Z}_2$, where $H^1(F, \mathbb{Z}_2)^\varphi$ denotes the subgroup of φ–invariant classes. In other words, the normal bundle is isomorphic to a tensor product $N = \widetilde{L} \otimes f^*Q$, where $\widetilde{L}$ is the line bundle induced on the mapping torus from a $\mathbb{Z}$–equivariant line bundle L on F, and Q is a line bundle on S^1_λ. Choosing a $\mathbb{Z}$–invariant flat connection on L and a flat connection on Q, we obtain a flat connection ∇ on N. The Poisson module structure on N is then simply

$$\partial = \pi \circ \nabla + \partial_t,$$

and so the residue χ of the Poisson line bundle is $(\omega^{n-1} \wedge f^*dt)^{-1}$. The constant λ retained in the construction has an invariant meaning: it is the ratio of the volume of D (with respect to χ^{-1}) to the volume of the symplectic leaf F (with respect to ω^{n-1}). Summarizing the above discussion, we obtain the following result.

Proposition 4.1.11. *The linearization of a proper log symplectic $2n$–manifold along a connected component D of its degeneracy locus is classified up to isomorphism by the following data: a compact symplectic $(2n-2)$–manifold (F, ω), a symplectomorphism φ, a cohomology class in $H^1(F, \mathbb{Z}_2)^\varphi \times \mathbb{Z}_2$, and a positive real number λ, called the modular period.*

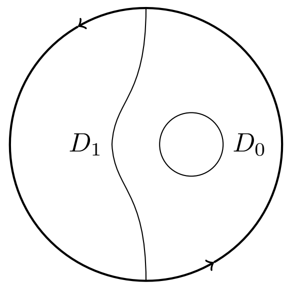
One of the main results of [16], extending the result in [34] for surfaces, is a proof, via a Moser-type deformation argument, that log symplectic manifolds are linearizable, namely that a tubular neighbourhood of each component D_j of the degeneracy locus is isomorphic, as a log symplectic manifold, to a neighbourhood of the zero section in the linearization along D_j.

Theorem 4.1.12 (Guillemin–Miranda–Pires [16])**.** *A log symplectic manifold is linearizable along its degeneracy locus.*

Example 4.1.13. The cubic polynomial $g(x) = x(x-1)(x-t)$, $0 < t < 1$, defines a Poisson structure on $\mathbb{R}^2$ given by

$$\pi = (g(x) - y^2)\partial_x \wedge \partial_y,$$

which extends smoothly to a log symplectic structure on $\mathbb{R}P^2$ with degeneracy locus D given by the real elliptic curve $y^2 = g(x)$, as shown below.



The degeneracy locus has two connected components: D_0, containing $\{(0,0),(t,0)\}$ and with trivial normal bundle, and D_1, containing $\{(1,0),(\infty,0)\}$ and with nontrivial normal bundle. The residue χ of π along D is such that χ^{-1} coincides with the Poincaré residue

$$\frac{dx}{2y} = \frac{dx}{2\sqrt{x(x-1)(x-t)}}.$$

This extends to a holomorphic form on the complexified elliptic curve, in which D_0, D_1 are cohomologous, so that the modular periods λ_0, λ_1 of D_0, D_1 must coincide. The modular period λ_0 is therefore a classical elliptic period [20], given by the Gauss hypergeometric function

$$\lambda_0(t) = \pi F(\tfrac{1}{2}, \tfrac{1}{2}, 1; t).$$

4.2 Log symplectic surfaces

The log symplectic structures on an orientable surface were classified by Radko [34]. In this section, we recall these results and provide a classification for log symplectic structure on an non-orientable surface.

Let (Σ, π) be a log symplectic surface, and let $D_\pi = \coprod_{j=1}^{n} \gamma_j$ be the degeneracy locus where γ_j are loops. Recall the modular period of γ_j is

$$\lambda_j(\pi) \coloneqq \int_{\gamma_j} {\chi_j}^{-1} \tag{4.2.1}$$

where χ_j is residue covolume form of γ_j as in Proposition 4.1.8. Let $\omega = \pi^{-1}$ be the inverse of π, which is a well-defined symplectic structure on $\Sigma \setminus D$. Let π_0 be a non-degenerate Poisson structure on Σ, and we write $\pi = h\pi_0$ where h is smooth function vanishes transversally on D_π. The ***regularized volume*** of (Σ, π)

$$V_\pi \coloneqq \lim_{\epsilon \to 0} \int_{|h| > \epsilon} \omega. \tag{4.2.2}$$

is independent of the choice of π_0.

Theorem 4.2.1. *[34] For a compact connected orientable surface Σ, two log symplectic structures, π and π', are equivalent by an orientation-preserving Poisson isomorphism, if and only if the following are satisfied:*

1. *$(\Sigma, D_\pi) \simeq (\Sigma, D_{\pi'})$. That is, there is a diffeomorphism $\varphi : \Sigma \to \Sigma$ such that $\varphi(D_\pi) = D_{\pi'}$, say $\varphi(\gamma_j) = \gamma'_j$.*
2. *The corresponding modular periods are equal, i.e. $\lambda_j(\pi) = \lambda_j(\pi')$.*
3. *The regularized volumes are the same, i.e. $V(\pi) = V(\pi')$.*

Remark 4.2.2. If $(\Sigma, D_\pi) \simeq (\Sigma, D_{\pi'})$ and $\lambda_j(\pi) = \lambda_j(\pi')$, i.e. the first two conditions are satisfied, and we have $V_\pi = -V_{\pi'}$, then π and π' are equivalent by an orientation-reversing Poisson isomorphism.

From Theorem 4.2.1, we may completely characterize the moduli space of log symplectic structures on an orientable surface by a labeled graph as follows. [3]

For a log symplectic surface (Σ, π) with degeneracy locus $D = \coprod_{j=1}^{n} \gamma_j$, we enumerate the open symplectic leaves: $\Sigma \setminus D = \coprod_{i=1}^{m} V_i$. We represent each open symplectic leaf V_i by a vertex, and represent each γ_j by an edge connecting the two adjacent open symplectic leaves. We label each edge with the corresponding modular period λ_j of γ_j, and label each vertex with the genus g_i of the corresponding symplectic leaf V_i. Furthermore, we label the entire graph with the absolute value of the regularized volume $V(\pi)$ of (Σ, π).

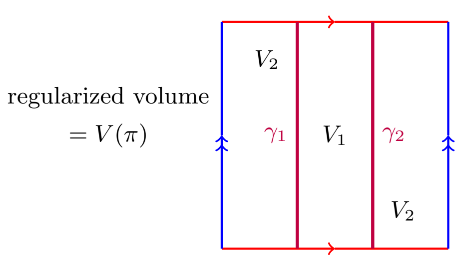
Then Theorem 4.2.1 together with Remark 4.2.2 implies that the moduli space of log symplectic structures on a compact orientable surface Σ of genus g is characterized by the following a combinatorial data:

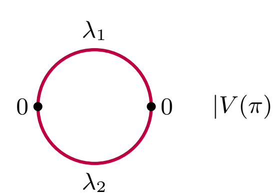
1. A bipartite graph [3] Γ with m vertices and n edges such that each vertex is labeled by a non-negative integer g_i, and each edge is labeled by a positive real number λ_j.
2. The sum of the integers g_i and the first betti number b_1 of the graph Γ equals g, i.e.

$$g = \sum_{i=1}^{m} g_i + b_1. \tag{4.2.3}$$

3. A positive real number $V(\pi)$.

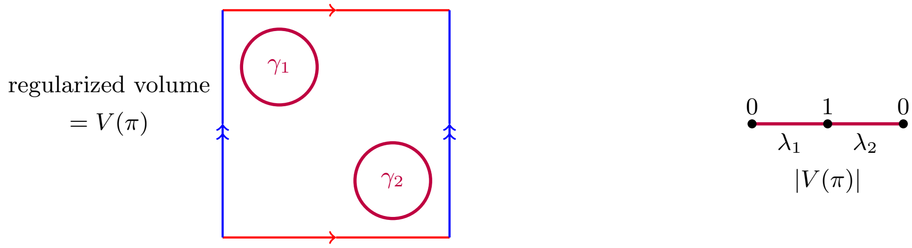
Example 4.2.3. For a torus $\mathbb{T}^2$, let us consider a log symplectic structure π on $\mathbb{T}^2$ such that the degeneracy locus D_π has exactly two homotopic circles. The torus is illustrated on the left and its labeled graph is on the right:





[3]A graph is bipartite if each cycle has even number of vertices.

Let us consider another log symplectic structure π' on $\mathbb{T}^2$ such that the degeneracy locus $D_{\pi'}$ has exactly two homotopically trivial circles. The torus is illustrated on the left and its labeled graph is on the right:



□

Now, we turn our attention to the log symplectic structures on a compact non-orientable surface.

Lemma 4.2.4. *Let (Σ, π) be a compact connected non-orientable log symplectic surface, and let $p : \widetilde{\Sigma} \to \Sigma$ be the orientable double cover. There is a unique log symplectic structure $\widetilde{\pi}$ on $\widetilde{\Sigma}$ such that $p_*(\widetilde{\pi}) = \pi$. Furthermore, the regularized volume of $\widetilde{\pi}$ vanishes, i.e. $V_{\widetilde{\pi}} = 0$.*

Proof. Since $p : \widetilde{\Sigma} \to \Sigma$ is a local diffeomorphism and being log symplectic is a local condition, it follows that the unique Poisson structure $\widetilde{\pi}$ on Σ such that $p_*(\widetilde{\pi}) = \pi$ is log symplectic. On the other hand, $\sigma_*(\widetilde{\pi}) = -\widetilde{\pi}$ where $\sigma : \widetilde{\Sigma} \to \widetilde{\Sigma}$ is the involution of the orientable cover, so we have $V_{\widetilde{\pi}} = 0$. □

Proposition 4.2.5. *For a compact connected non-orientable surface Σ, two log symplectic structures, π and π', are equivalent by a Poisson isomorphism, if and only if the following are satisfied:*

1. *$(\Sigma, D_\pi) \simeq (\Sigma, D_{\pi'})$. That is, there is a diffeomorphism $\varphi : \Sigma \to \Sigma$ such that $\varphi(D_\pi) = D_{\pi'}$, say $\varphi(\gamma_j) = \gamma'_j$, for $j = 1, \ldots, n$.*
2. *The corresponding modular periods are equal, i.e. $\lambda_j(\pi) = \lambda_j(\pi')$.*

Sketch of proof. Let $\widetilde{\varphi} : (\widetilde{\Sigma}, D_{\widetilde{\pi}}) \to (\widetilde{\Sigma}, D_{\widetilde{\pi}'})$ be the lift of $\varphi : (\Sigma, D_\pi) \to (\Sigma, D_{\pi'})$. By Lemma 4.2.4, we have $V_{\widetilde{\pi}} = V_{\widetilde{\pi}'}$. By Theorem 4.2.1, we obtain a Poisson isomorphism $\widetilde{\psi} : (\widetilde{\Sigma}, \widetilde{\pi}) \to (\widetilde{\Sigma}, \widetilde{\pi}')$.

The proof of Theorem 4.2.1 uses Moser's trick to obtain diffeomorphisms $\rho_t : \widetilde{\Sigma} \to \widetilde{\Sigma}$ for $0 \leq t \leq 1$ such that $\rho_0 = \mathsf{id}_{\widetilde{\Sigma}}$ and $\rho_1 = \widetilde{\psi}$. We observe that the Poisson maps ρ_t may be chosen to be equivariant with respect to the orientable cover, and therefore descends to a Poisson isomorphism $\psi : (\Sigma, \pi) \to (\Sigma, \pi')$. □

Proposition 4.2.5 yields a complete characterization of the moduli space of log symplectic structures on a compact non-orientable surface by a labeled graph with half edges. Similar to the case of orientable surfaces, we represent each open symplectic leaf V_i by a vertex. If a degeneracy circle γ_j has an orientable normal bundle, then we represent γ_j by an edge connecting the two adjacent open symplectic leaves; If a degeneracy circle γ_j has a non-orientable normal bundle, then we represent γ_j by a half-edge connected to the adjacent open symplectic leaf. We label each edge (or half-edge) γ_j with its modular period λ_j; and we label each vertex with genus g_i of the corresponding symplectic leaf V_i.

For a graph with half-edges Γ, there is a regular graph $\widetilde{\Gamma}$ which is a double cover over Γ. Recall that a compact non-orientable surface is either isomorphic to the projective plane $\mathbb{RP}^2$ with n handles or the Klein bottle with n handles. Proposition 4.2.5 implies that the moduli space of log symplectic structures on a compact non-orientable surface Σ is characterized by the following a combinatorial data:

1. A bipartite graph with half edges Γ such that each vertex is labeled by a non-negative integer g_i, and each edge (or half-edge) is labeled by a positive real number λ_j.

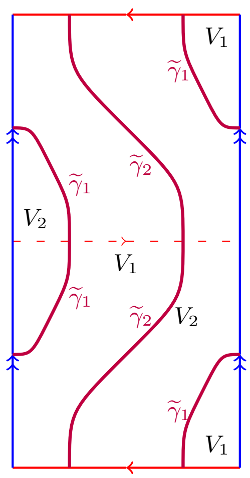
2. The sum of twice the integers g_i and the first betti number $\widetilde{b}_1$ of the double cover graph $\widetilde{\Gamma}$ is determined by the underlying compact non-orientable surface Σ. That is, if we define

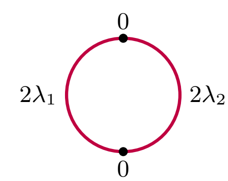
$$\widetilde{g} \coloneqq \sum_{i=1}^{m} 2g_i + \widetilde{b}_1, \tag{4.2.4}$$

then $\widetilde{g} = 2n$ if Σ is isomorphic to $\mathbb{RP}^2$ with n handles, and $\widetilde{g} = 2n+1$ if Σ is isomorphic to a Klein bottle with n handles.

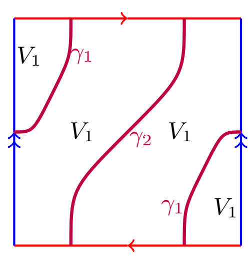
Example 4.2.6. For a Klein bottle K^2, let us consider a log symplectic structure π on K^2 such that the degeneracy locus D_π has exactly two homotopic circles with non-orientable normal bundles. The Klein bottle with its orientable double cover is illustrated on the left, and the labeled graph with its double cover is on the right:

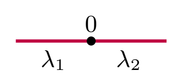
Orientable cover: $\mathbb{T}^2$





Klein bottle: K^2





□

For a compact orientable log symplectic surface (Σ, π), the Poisson cohomology $H^*_\pi(\Sigma, \pi)$ was computed in [34], using the Mayer-Vietoris sequence of an open cover $\{U, V\}$ where U is a neighbourhood of the degeneracy locus and V is the complement of the degeneracy locus. We summarize the results here:

$$\begin{aligned}
H^0_\pi(\Sigma) &= <1>, \\
H^1_\pi(\Sigma) &= <Z_1, \cdots, Z_n> \oplus \pi^\sharp\left(H^1_{\mathtt{deR}}(\Sigma)\right), \\
H^2_\pi(\Sigma) &= <\pi_0, \pi_1, \cdots, \pi_n>,
\end{aligned}$$

where Z_j is the transverse Poisson vector field on the degeneracy circle γ_j defined in (4.1.1); π_0 is some non-degenerate Poisson structure on Σ; and $\pi_j = f_j \pi$ where f_j is a bump function around γ_j.

Using the same strategy, for a compact non-orientable log symplectic surface (Σ, π), the Poisson cohomology $H^*_\pi(\Sigma, \pi)$ is as follows:

$$\begin{aligned} H^0_\pi(\Sigma) &= <1>, \\ H^1_\pi(\Sigma) &= <Z_1, \cdots, Z_n> \oplus \pi^\sharp \left(H^1_{\mathtt{deR}}(\Sigma)\right), \\ H^2_\pi(\Sigma) &= <\pi_1, \cdots, \pi_n> . \end{aligned}$$

Comparing to the case of orientable surfaces, the only difference is that we lose $<\pi_0>$ in $H^2_\pi(\Sigma)$ because we cannot deform the regularized volume.

Chapter 5

Examples of birational constructions

In this chapter, we apply the birational construction introduced in §3 to several example of Lie groupoids whose Lie algebroids are lower elementary modifications of the tangent algebroid.

In §5.1, we construct the adjoint groupoid of an log tangent algebroid. In the real case, this is the algebroid whose sections are the vector fields tangent to a collection of normal crossing hypersurfaces. In this case, the construction reproduces the 'puff' construction of Monthubert [30]. In the complex case, we restrict our attention to the case of Riemann surface with a divisor. The log tangent algebroid has sections which are vector fields vanishing up to the prescribed order with respect to the divisor. In §5.2, we construct the adjoint symplectic groupoid of a log symplectic manifold following [12].

5.1 Th log pair groupoid

5.1.1 Normal crossing hypersurfaces

We slightly generalize Definition 4.1.2 by allowing D to be a normal crossing divisor as in Definition 3.3.1. That is, let M be a smooth manifold, and let $D = L_1 + L_2 + \ldots + L_n$ where L_j are closed normal crossing hypersurfaces. The log tangent bundle $T(M, D)$ is the vector bundle whose sheaf of sections are vector fields tangent to each L_j.

The log tangent bundle $T(M, D)$ is a Lie algebroid, and can be written as a fiber product of Lie algebroids as follows:

$$[TM\!:\!TL_1, TL_2, \ldots, TL_n] = [TM\!:\!TL_1] \oplus_{TM} [TM\!:\!TL_2] \oplus_{TM} \ldots \oplus_{TM} [TM\!:\!TL_n].$$

For simplicity, let us assume that each L_j are connected. Let $\mathcal{G} = \mathrm{Pair}(M)$ be the pair groupoid of M. For each j, let $\mathcal{H}_j = \mathrm{Pair}(L_j)$ be the pair groupoid of L_j, and let $\mathcal{G}_j = [\mathcal{G}\!:\!\mathcal{H}_j]^c$ be the source-connected blow-up groupoid as in Theorem 3.1.2. By Proposition 3.3.6, the ***log pair groupoid***

$$\begin{aligned}
&\mathrm{Pair}(M, D) = \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \times_{\mathcal{G}} \ldots \times_{\mathcal{G}} \mathcal{G}_n \\
&\quad = [\mathrm{Pair}(M)\!:\!\mathrm{Pair}(L_1)]^c \times_{\mathrm{Pair}(M)} [\mathrm{Pair}(M)\!:\!\mathrm{Pair}(L_2)]^c \times_{\mathrm{Pair}(M)} \ldots \times_{\mathrm{Pair}(M)} [\mathrm{Pair}(M)\!:\!\mathrm{Pair}(L_n)]^c
\end{aligned}$$

integrates $T(M, D)$, i.e. the 'puff' construction by Monthubert [30]. Note $\mathrm{Pair}(M, D)$ is source-connected.

By Proposition 3.3.8, $\mathrm{Pair}(M, D)$ may also be obtained by iterative blow-up's. Inductively applying Remark 3.1.8, we conclude that $\mathrm{Pair}(M, D)$ is the adjoint integration of $T(M, D)$.

5.1.2 Divisors on a Riemann surface

For this subsection, we work in the complex category, where the groupoid $[\mathcal{G} : \mathcal{H}]$ is constructed by a complex blow-up.

Definition 5.1.1. Let X be a Riemann surface, and let $\mathsf{D} = d_1p_1 + d_2p_2 + \ldots + d_kp_k$ be a divisor on X. We define the log tangent sheaf twisted by $-\mathsf{D}$, denoted by $\mathcal{T}\mathsf{X}(-\mathsf{D})$, to be the sheaf of vector fields on X vanishing up to order d_i at the point p_i.

In particular, if D is trivial, then we recover the tangent sheaf $\mathcal{T}\mathsf{X}$.

The log tangent sheaf $\mathcal{T}\mathsf{X}(-\mathsf{D})$ is locally free. We denote the corresponding vector bundle by $T\mathsf{X}(-\mathsf{D})$. Equipped with the bracket of vector fields, $T\mathsf{X}(-\mathsf{D})$ is a Lie algebroid.

The representations of these log tangent algebroids are the meromorphic connections. In this case, the equivalence between the representation of the Lie algebroid and the representation of its ssc integration is essentially the Riemann-Hilbert correspondence. If the divisor contains higher order poles, then the representation of the log tangent algebroid exhibit the Stoke's phenomenon. For more details, please see [13] and its references.

Consider the pair groupoid $\mathrm{Pair}(\mathsf{X}) \rightrightarrows X$, we do a complex blow-up along $\mathsf{id}(p_1) \subset \mathrm{Pair}(X)$ and obtain the blow-up groupoid

$$\mathrm{Pair}(\mathsf{X}, p_1) = [\mathrm{Pair}(\mathsf{X}) : \mathsf{id}(p_1)] \rightrightarrows X$$

as in Theorem 3.1.2, which integrates the Lie algebroid $T\mathsf{X}(-p_1)$. Notice that $\mathsf{id}(p_1) \subset \mathrm{Pair}(\mathsf{X}, p_1)$ is, again, a subgroupoid of complex codimension 2, so we may do an iterated blow-up and obtain $\mathrm{Pair}(\mathsf{X}, 2p_1) = [\mathrm{Pair}(\mathsf{X}, p_1) : \mathsf{id}(p_1)]$. Proceed in this fashion, we have $\mathrm{Pair}(\mathsf{X}, d_1p_1)$ which integrates $T\mathsf{X}(-d_1p_1)$.

Similarly, for each $j = 1, \ldots, k$, we may obtain $\mathrm{Pair}(\mathsf{X}, d_jp_j)$ integrating $T\mathsf{X}(-d_jp_j)$. By Proposition 3.3.6, the fiber product product

$$\mathrm{Pair}(\mathsf{X}, \mathsf{D}) = \mathrm{Pair}(\mathsf{X}, d_1p_1) \times_{\mathrm{Pair}(\mathsf{X})} \mathrm{Pair}(\mathsf{X}, d_2p_2) \times_{\mathrm{Pair}(\mathsf{X})} \ldots \times_{\mathrm{Pair}(\mathsf{X})} \mathrm{Pair}(\mathsf{X}, d_kp_k) \tag{5.1.1}$$

integrates the log tangent algebroid $T\mathsf{X}(-\mathsf{D})$. In addition, $\mathrm{Pair}(\mathsf{X}, \mathsf{D})$ is adjoint by Proposition 3.3.8 and Remark 3.1.8.

5.2 The symplectic pair groupoid

In this section, we construct the adjoint symplectic groupoid of a log symplectic manifold, following [12].

Let (M, π) be a proper log symplectic manifold. The degeneracy locus $D = \coprod_{j \in \mathsf{D}} D_j$ is a closed hypersurface where D_j are the connected components. We shall prove that beginning with the Poisson pair groupoid $(\mathrm{Pair}(M), \pi \oplus -\pi)$, the log pair groupoid $\mathrm{Pair}(M, D)$ is a Poisson groupoid too. For this, we apply Theorem 3.2.3 in the following special case.

Proposition 5.2.1. *Let (M_1, π_1) and (M_2, π_2) be log symplectic manifolds with degeneracy loci D_1 and D_2, respectively. Then there is a unique Poisson structure $\widetilde{\pi}$ on the blowup $\mathsf{Bl}_{D_1 \times D_2}(M_1 \times M_2)$ such that*

$p_*(\widetilde{\pi}) = \pi_1 \oplus \pi_2$, and $\widetilde{\pi}$ is log symplectic on

$$\widetilde{X} := \mathsf{Bl}_{D_1\times D_2}(M_1 \times M_2)\backslash(\overline{\overline{M_1 \times D_2}} \cup \overline{\overline{D_1 \times M_2}}),$$

with degeneracy locus given by the exceptional divisor $\widetilde{X} \cap \mathbb{P}\left(N(D_1 \times D_2)\right)$ in $\widetilde{X}$.

Proof. Let $M = M_1 \times M_2$, and note that the transverse Poisson structure π_N along $L = D_1 \times D_2$ vanishes. By Theorem 3.2.3, there is a unique Poisson structure $\widetilde{\pi}$ on $\mathsf{Bl}_L(M)$ such that $p_*(\widetilde{\pi}) = \pi_1 \oplus \pi_2$. We must show that the Pfaffian of $\widetilde{\pi}$ vanishes transversely along $\mathsf{Bl}_L(M) \cap \mathbb{P}(NL)$.

By the linearization Theorem 4.1.12, we may, as in Proposition 4.1.9, take $M_i = \mathrm{tot}(N_i)$ to be the total space of a line bundle N_i over D_i, and after choosing connections we obtain

$$\pi_i = \sigma_i + Z_i \wedge E_i,$$

where Z_i and σ_i are the vector and bivector fields on D_i, respectively, horizontally lifted to M_i, and E_i is the Euler vector field on $\mathrm{tot}(N_i)$, which may be viewed as a section $E_i \in \Gamma(D_i, N^*D_i \otimes ND_i)$.

Then, if $\dim M_i = 2n_i$, the Pfaffian of $\pi = \pi_1 \oplus \pi_2$ is

$$\begin{aligned} \pi^{n_1+n_2} &= cE_1 \wedge E_2 \wedge Z_1 \wedge \sigma_1^{n_1-1} \wedge Z_2 \wedge \sigma_2^{n_2-1} \\ &= cE_1 \wedge E_2 \wedge \chi_1 \wedge \chi_2, \end{aligned} \tag{5.2.1}$$

where c is a nonzero constant and $\chi_i \in \Gamma(D_i, \wedge^{n_i} TD_i)$ are the residues from Definition 4.1.6. Hence we see that the Pfaffian may be viewed invariantly as a nonvanishing section

$$\pi^{n_1+n_2} \in \Gamma(L, \mathrm{Sym}^2 N^*L \otimes \wedge^2 NL \otimes \det TL).$$

After blowing up, this section defines a covolume form which vanishes linearly on the exceptional divisor, in the following way. First, the blowup $\mathsf{Bl}_L(M)$ may be identified with the total space of the tautological line bundle $q : U \to \mathbb{P}(NL)$. Covolume forms on $\mathrm{tot}(U)$ which vary linearly on the fibers are sections of

$$q^*U^* \otimes \det(T(\mathrm{tot}(U))) = q^* \det T\mathbb{P}(NL). \tag{5.2.2}$$

Using the blow-down map $p : \mathbb{P}(NL) \to L$, we write the Euler sequence

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow U^* \otimes p^*NL \longrightarrow V \longrightarrow 0\ , \tag{5.2.3}$$

defining the relative tangent bundle V for the projection p. Also, we have the exact sequence

$$0 \longrightarrow V \longrightarrow T\mathbb{P}(NL) \longrightarrow p^*TL \longrightarrow 0\ . \tag{5.2.4}$$

Combining (5.2.2), (5.2.3), and (5.2.4), we see that fiberwise linear covolume forms on $\mathrm{tot}(U)$ are given by sections of

$$\det(U^* \otimes p^*NL) \otimes \det p^*TL = (U^*)^2 \otimes p^*(\wedge^2 NL \otimes \det TL),$$

where we have used the fact that L is codimension 2.

Squaring the restriction $p^*N^* \to U^*$, we obtain a natural map $r : p^*\mathrm{Sym}^2 N^*L \to (U^*)^2$, so that the

Pfaffian defines a section

$$r \otimes (\pi^{n_1+n_2}) \in \Gamma(\mathbb{P}(NL), (U^*)^2 \otimes p^*(\wedge^2 NL \otimes \det TL)).$$

Therefore, after blow-up, the Pfaffian defines a fiberwise linear covolume form on $\mathrm{tot}(U)$ which varies quadratically along the projective fibers, vanishing (due to the factorization (5.2.1)) along the fibers over the pair of sections $\mathbb{P}(N(M_1 \times D_2))$, $\mathbb{P}(N(D_1 \times M_2))$ of $\mathbb{P}(NL)$, which coincide with the loci $\overline{\overline{M_1 \times D_2}}$ and $\overline{\overline{D_1 \times M_2}}$ along the exceptional divisor, as required. □

Theorem 5.2.2. *Let (M, π) be a log symplectic manifold with degeneracy locus D, and let $p : \mathrm{Pair}(M, D) \to \mathrm{Pair}(M)$ be the blow-down groupoid morphism. Then there is a unique log symplectic structure σ on $\mathrm{Pair}(M, D)$ such that $p_*(\sigma) = -\pi \oplus \pi$. This makes $(\mathrm{Pair}(M, D), \sigma)$ a Poisson groupoid over (M, π), and the blow-down a morphism of Poisson groupoids.*

Proof. Since $D_j \cap D_k = \varnothing$ for $j \neq k$, we have

$$\mathrm{Pair}(M, D) = [\mathrm{Pair}(M) : \mathrm{Pair}^c(D)] = \mathsf{Bl}_{\mathrm{Pair}^c(D)}(\mathrm{Pair}(M)) \backslash \overline{\overline{s^{-1}(D)}} \cup \overline{\overline{t^{-1}(D)}},$$

where $\mathrm{Pair}^c(D) = \coprod_{j \in \mathtt{D}} \mathrm{Pair}(D_j)$. It follows that the proper transforms of

$$D \times M = s^{-1}(D), \quad M \times D = t^{-1}(D)$$

have been removed, so by Proposition 5.2.1, we obtain the required log symplectic structure σ on $\mathrm{Pair}(M, D)$ lifting $-\pi \oplus \pi$ on $\mathrm{Pair}(M)$.

Since the graph of the multiplication on $\mathrm{Pair}(M, D)$ is coisotropic on an open dense subset, it follows that the graph must be everywhere coisotropic, proving $(\mathrm{Pair}(M, D), \sigma)$ is a Poisson groupoid. □

To obtain the adjoint symplectic groupoid of the proper log symplectic manifold (M, π), we need to do a second blow-up to the log pair groupoid $\mathrm{Pair}(M, D)$. By Theorem 4.1.12, each connected component D_j of the degeneracy locus D is a symplectic fiber bundle $f_j : D_j \to \gamma_j$ over a circle $\gamma_j \cong S^1$. With $\gamma = \coprod_{j \in \mathtt{D}} \gamma_j$, we obtain a projection map

$$f : D \to \gamma,$$

which endows the pair (M, D) with a structure akin to that of a manifold with fibered boundary [28]. The Lie algebroid $T^*_\pi M$ of such a Poisson structure is isomorphic via the anchor map to the Lie algebroid $T_f(M, D)$ whose sheaf of sections is given by

$$T_f(M, D)(U) = \{X \in \Gamma(U, TU) \mid X|_D \in \Gamma(D, \ker Tf)\},$$

and this algebroid may be expressed as an elementary modification in the sense of Definition 3.1.4, as follows. The log tangent algebroid $T(M, D)$ restricts to D to define a Lie algebroid $T(M, D)|_D$, naturally isomorphic to the Atiyah algebroid $\mathtt{At}(ND)$ of infinitesimal symmetries of the normal bundle of D. The composition of the projection in the exact sequence

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathtt{At}(ND) \longrightarrow TD \longrightarrow 0$$

with $Tf : TD \to T\gamma$ has a kernel $\mathtt{At}_f(ND) \subset \mathtt{At}(ND)$, which may be viewed as a relative Atiyah algebroid with respect to the fibration f. Therefore we obtain the following representation of the Lie algebroid underlying $T^*_\pi M$ as an elementary modification:

$$T_f(M,D) = [T(M,D) : \mathtt{At}_f(ND)]. \tag{5.2.5}$$

By construction, $\mathtt{At}_f(ND)$ is a subalgebroid of $T(M,D)$, and we now construct the corresponding subgroupoid of $\mathrm{Pair}(M,D)$, which may be viewed as the gauge groupoid of ND relative to the fibration f.

Inside the pair groupoids $\coprod_{j\in\mathtt{D}} \mathrm{Pair}(D_j)$, we have the pair groupoids relative to f, given by

$$\mathrm{Pair}_f(D) = \coprod_{j\in\mathtt{D}} (D_j \times_{\gamma_j} D_j).$$

Its preimage in the exceptional divisor for the blow-down $p : \mathrm{Pair}(M,D) \to \mathrm{Pair}(M)$ is

$$\mathtt{GL}^+_f(ND) = p^{-1}\left(\coprod_{j\in\mathtt{D}} (D_j \times_{\gamma_j} D_j)\right). \tag{5.2.6}$$

This defines a Poisson subgroupoid $\mathtt{GL}^+_f(ND) \rightrightarrows D$, with Lie bialgebroid $(\mathtt{At}_f(ND), TD)$, which is a codimension 2 symplectic leaf in the log symplectic manifold $\mathrm{Pair}(M,D)$. We now perform a blow-up to obtain a symplectic groupoid integrating $T^*_\pi M$.

Theorem 5.2.3. *Let (M,π) be a proper log symplectic manifold with degeneracy locus $D \subset M$, and let $\mathtt{GL}^+_f(ND) \rightrightarrows D$ be the subgroupoid of $\mathrm{Pair}(M,D) \rightrightarrows M$ defined in (5.2.6). Then the groupoid*

$$\mathrm{Pair}_\pi(M) = [\mathrm{Pair}(M,D) : \mathtt{GL}^+_f(ND)] \tag{5.2.7}$$

has a unique symplectic structure ω such that the blow-down $\mathrm{Pair}_\pi(M) \to \mathrm{Pair}(M,D)$ is Poisson. This makes $(\mathrm{Pair}_\pi(M),\omega)$ a symplectic groupoid integrating (M,π), and the blow-down a morphism of Poisson groupoids. *a symplectic groupoid integrating (M,π), and the blow-down a morphism of Poisson groupoids.*

Proof. Let $(\mathrm{Pair}(M,D),\sigma) \rightrightarrows (M,\pi)$ be the Poisson groupoid constructed in Theorem 5.2.2, and let s,t be its source and target maps. The subgroupoid $\mathtt{GL}^+_f(ND) \rightrightarrows D$ is a symplectic leaf in the degeneracy locus of the log symplectic manifold $(\mathrm{Pair}(M,D),\sigma)$. It follows that the transverse Poisson structure on the normal bundle of $\mathtt{GL}^+_f(ND)$ is degenerate, so by Theorem 3.2.3, the blow-up

$$\mathsf{Bl}_{\mathtt{GL}^+_f(ND)}(\mathrm{Pair}(M,D)) \xrightarrow{p'} \mathrm{Pair}(M,D)$$

inherits a unique Poisson structure $\widetilde{\sigma}$ such that $p'_*\widetilde{\sigma} = \sigma$. Furthermore, since $\widetilde{\sigma}$ is multiplicative away from the exceptional divisor, it must be multiplicative by continuity.

Recall that for the pair Poisson groupoid, we have its Lie bialgebroid

$$\mathtt{Lie}\,(\mathrm{Pair}(M), \pi \oplus -\pi) = (TM, T^*_\pi M) = (TM, T_f(M,D)).$$

By Theorem 3.2.11, we have

$$\begin{aligned}\texttt{Lie}\,(\mathrm{Pair}(M,D),\sigma) &= ([TM\!:\!TD],\{T_f(M,D)\!:\!TD\}) \\ &= (T(M,D),T(M,D)), \\ \texttt{Lie}\,(\mathrm{Pair}_\pi(M),\widetilde{\sigma}) &= ([T(M,D)\!:\!\texttt{At}_f(ND)],\{T(M,D)\!:\!TD\}) \\ &= (T_f(M,D),TM) = (T^*_\pi M,TM).\end{aligned}$$

By Remark 2.2.10, $\big(\mathrm{Pair}_\pi(M),\omega=\widetilde{\sigma}^{-1}\big)$ is a symplectic groupoid of (M,π).

□

Remark 5.2.4. The blow-up of a source-connected Lie groupoid along a source-connected subgroupoid may fail to be source-connected, since the exceptional divisor in (3.1.2) consists of a projective bundle with two families of hyperplanes removed, and the complement of a pair of hyperplanes in $\mathbb{R}P^n$ is generically disconnected. In the case of the symplectic pair groupoid (5.2.7), however, the deleted loci coincide, and so $\mathrm{Pair}_\pi(M)$, as defined, is source-connected.

In particular, since the log pair groupoid $\mathrm{Pair}(M,D)$ is adjoint, by Remark 3.1.8, the symplectic pair groupoid $(\mathrm{Pair}_\pi(M),\omega)$ is also adjoint.

Using Theorem 3.1.9, we may also construct the symplectic pair groupoid $\mathrm{Pair}_\pi(M)$ by a single blow-up.

Corollary 5.2.5. *The symplectic pair groupoid* $(\mathrm{Pair}_\pi(M),\omega)$ *may be alternatively constructed as* $[\mathrm{Pair}(M)\!:\!\mathrm{Pair}_f(D)]^c$.

Proof. Recall that $\mathrm{Pair}_f(D)\subset\mathrm{Pair}^c(D)\subset\mathrm{Pair}(M)$ and the symplectic pair groupoid $\mathrm{Pair}_\pi(M)$ is constructed as

$$\begin{aligned}\mathrm{Pair}_\pi(M) &= [\mathrm{Pair}(M,D)\!:\!p^{-1}(\mathrm{Pair}_f(D))] \\ &= [[\mathrm{Pair}(M)\!:\!\mathrm{Pair}^c(D)]^c\!:\!p^{-1}(\mathrm{Pair}_f(D))] \\ &= [[\mathrm{Pair}(M)\!:\!\mathrm{Pair}^c(D)]^c\!:\!\overline{\overline{\mathrm{Pair}_f(D)}}].\end{aligned}$$

where $p:\mathrm{Pair}(M,D)\to\mathrm{Pair}(M)$ is the blow-down map. Since $\mathrm{Pair}_\pi(M)$ is source-connected, by Theorem 3.1.9, we obtain $\mathrm{Pair}_\pi(M)=[\mathrm{Pair}(M)\!:\!\mathrm{Pair}_f(D)]^c$. □

Chapter 6

Gluing construction and groupoid classifications

In this chapter, we present a general method for constructing Lie groupoids on a manifold by gluing together groupoids defined on the open sets of a covering. For this to be possible, the open cover must be adapted to the orbits of the lie algebroid in question; groupoids are inherently global objects and in general cannot be so easily decomposed. In §6.1, we consider a simple kind of cover, called an ***orbit cover***, which permits the result to hold.

We use this construction to explicitly describe the category of all groupoids integrating the log tangent bundle of a closed hypersurface $D \subset M$, as well as all Hausdorff symplectic groupoids integrating proper log symplectic manifolds in any dimension. This involves solving a local classification problem near each component of the degeneracy locus, and then combining these local results in a specific manner using a graph constructed from the global geometry of the manifold and its embedded hypersurfaces.

We also use this construction to explicitly describe the category of all groupoids integrating the Lie algebroid of meromorphic vector fields bounded by a divisor D on a Riemann surface X.

6.1 Orbit covers and the gluing of Lie groupoids

A typical way to construct manifolds is by the *fibered coproduct* operation, also known as gluing. If M_1, M_2 are manifolds equipped with open immersions $i_1 : U \hookrightarrow M_1$, $i_2 : U \hookrightarrow M_2$ from a manifold U, then the fibered coproduct is given by

$$M_1 \coprod_U M_2 = \frac{M_1 \coprod M_2}{i_1(x) \sim i_2(x)\ \forall x \in U}.$$

The caveat is that the resulting space is only Hausdorff when the graph of the equivalence relation above is closed in $M_1 \times M_2$. So, the fibered coproduct of manifolds is a possibly non-Hausdorff manifold.

Suppose that A is a Lie algebroid over a fibered coproduct of manifolds as above. We would like to construct a Lie groupoid integrating A by gluing integrations over M_1 and M_2 using open immersions of an integration over U. But, groupoids are non-local, and such a simple gluing construction is not generally possible. For example, the fundamental groupoid of the fibered coproduct should contain paths joining points in M_1 with points in M_2, and these may not be present in either of the fundamental groupoids of M_1, M_2. In general, a groupoid coproduct operation is required, whereby compositions absent from the naive gluing of spaces are formally adjoined. However, we are able to avoid this complication, by using a decomposition of the base which is adapted to the orbits of the Lie algebroid. Heuristically, we are able to naively glue groupoids, but only along interfaces where they are actually local over the base.

Essentially the same strategy was used by Nistor [31] to obtain groupoids of interest in the theory of pseudodifferential operators; the setup we present below, while less general than his theory of A-invariant stratifications, is well-adapted for our purpose, which is to classify integrations of log tangent and log symplectic algebroids.

Definition 6.1.1. Let A be a Lie algebroid over the manifold M. An open cover $\{U_i\}_{i\in I}$ of M is called an *orbit cover* if each orbit of the Lie algebroid is completely contained in at least one of the open sets U_i.

Definition 6.1.2. Given a groupoid $\mathcal{G} \rightrightarrows M$ and an open set $U \subset M$, we define the *restriction* of the groupoid to U to be the Lie subgroupoid

$$\mathcal{G}|_U = s^{-1}(U) \cap t^{-1}(U).$$

If A is an integrable Lie algebroid, we define $\mathtt{R}_U$ to be the restriction functor from the category of source-connected integrations of A to that of $A|_U$:

$$\begin{aligned} \mathtt{R}_U : \mathtt{Gpd}(A) &\to \mathtt{Gpd}(A|_U) \\ (\mathcal{G}, \phi) &\mapsto ((\mathcal{G}|_U)^c, \phi|_U). \end{aligned}$$

Remark 6.1.3. If $S \subset M$ is a submanifold which is closed and A–invariant, meaning $a(A|_S) \subset TS$, then any Lie algebroid orbit $\mathcal{O}$ intersecting S must be contained in S, and so $s^{-1}(S) = t^{-1}(S)$ for any source-connected groupoid $\mathcal{G}$ integrating A. For this reason, the restriction

$$\mathcal{G}|_S = s^{-1}(S) \cap t^{-1}(S) = s^{-1}(S)$$

is a submanifold of $\mathcal{G}$, and so defines a source-connected Lie subgroupoid $\mathcal{G}|_S \rightrightarrows S$ of $\mathcal{G}$.

Theorem 6.1.4. *Let A be an integrable Lie algebroid over M, and let $\{U_i\}_{i\in I}$ be a locally finite orbit cover of M. For each $i \in I$, let $\mathcal{G}_i \rightrightarrows U_i$ be a source-connected Lie groupoid integrating $A|_{U_i}$, and for each $i, j \in I$, let $\mathcal{G}_{ij} = \mathtt{R}_{U_i\cap U_j}(\mathcal{G}_i)$ and*

$$\phi_{ij} : \mathcal{G}_{ij} \xrightarrow{\cong} \mathcal{G}_{ji}$$

be an isomorphism of integrations of $A|_{U_i\cap U_j}$, such that $\phi_{ii} = \mathsf{id}$, $\phi_{ij} = \phi_{ji}^{-1}$, $\phi_{ij}(\mathcal{G}_{ij}\cap\mathcal{G}_{ik}) = \mathcal{G}_{ji}\cap\mathcal{G}_{jk}$, and $\phi_{ki}\phi_{jk}\phi_{ij} = \mathsf{id}$ for all $i, j, k \in I$, on $\mathcal{G}_{ij} \cap \mathcal{G}_{ik}$. This defines an equivalence relation, whereby $x \sim \phi_{ij}(x)$ for all $x \in \mathcal{G}_{ij}$ and for all $i, j \in I$. Then, we have the following:

1. *The fibered coproduct of manifolds*

$$\mathcal{G} = \coprod_{i\in I} \mathcal{G}_i \Big/ \sim \qquad (6.1.1)$$

is a source-connected Lie groupoid integrating A, such that $\mathtt{R}_{U_i}(\mathcal{G}) = \mathcal{G}_i$.

2. *The inclusion morphisms $\iota_k : \mathcal{G}_k \hookrightarrow \mathcal{G},\ k \in I$ make $\mathcal{G}$ a groupoid coproduct, meaning that for any groupoid which receives compatible morphisms from $\{\mathcal{G}_i\}_{i\in I}$, these morphisms must factor through a uniquely defined morphism from $\mathcal{G}$.*

3. *Every source-connected groupoid integrating A is the fibered (manifold) coproduct of its restrictions to the orbit cover $\{U_i\}_{i\in I}$.*

Proof. The fibered coproduct $\mathcal{G}$ in (6.1.1) inherits submersions $s, t : \mathcal{G} \to M$ and the embedding $\mathsf{id} : M \to \mathcal{G}$ from the corresponding maps on the groupoids $\mathcal{G}_i$, by the universal property of coproducts.

In the same way, the inverse maps on each $\mathcal{G}_i$ glue to a map $i : \mathcal{G} \to \mathcal{G}$. To see that the multiplication maps m_i of each groupoid $\mathcal{G}_i$ glue to a map $m : \mathcal{G}{}_s\times_t\mathcal{G} \to \mathcal{G}$, we use the orbit cover property, as follows.

Let $(h, g) \in \mathcal{G}_i \times \mathcal{G}_j$ be a representative for an arbitrary point in $\mathcal{G}{}_s\times_t\mathcal{G}$, so that $s_i(h) = t_j(g) = x$. By the orbit cover property, there exists k such that the A-orbit of x lies in U_k. Now, $t_i(s_i^{-1}(x))$ coincides with the $A|_{U_i}$–orbit of x, since $\mathcal{G}_i$ is source-connected. Therefore, $t_i(s_i^{-1}(x))$ must be contained in U_k. But we have $t_i(s_i^{-1}(x)) \subset U_i \cap U_k$, so $s_i^{-1}(x)$ coincides with $s_{ik}^{-1}(x) \subset (\mathcal{G}_i|_{U_i\cap U_k})^c$, which is identified with the source fiber of $(\mathcal{G}_k|_{U_i\cap U_k})^c$ by ϕ_{ik}. Therefore, the element h has a representative in $\mathcal{G}_k$. Similarly, g has a representative in $\mathcal{G}_k$. Hence, we may use the given multiplication m_k on $\mathcal{G}_k$ to define m in a neighbourhood of (h, g).

The argument above also shows that $\mathcal{G}$ is source-connected, since the source fiber $s^{-1}(x)$ coincides with the source fiber of a subgroupoid $\mathcal{G}_k$ such that U_k contains the A–orbit of x, and $\mathcal{G}_k$ is source-connected.

Part *ii)* follows from the universal property of the manifold coproduct, together with the fact that $\varphi : \mathcal{G} \to \mathcal{G}'$ is a Lie groupoid morphism if and only if $\varphi \circ \iota_k$ is a Lie groupoid morphism for all $k \in K$.

Part *ii)* implies that any source-connected integration $\mathcal{G}$ receives an isomorphism from the coproduct of its restriction $\mathtt{R}_{U_i}(\mathcal{G})$, establishing *iii)*.

□

Example 6.1.5. Let $p \in S^1$ be a point on the circle, and $T(S^1, p)$ the Lie algebroid of vector fields vanishing at p. We may construct a Lie groupoid integrating this algebroid as follows. Express the circle as a fibered coproduct of $U = \mathbb{R}$, $V = \mathbb{R}$, with gluing map $\phi : U \setminus \{0\} \to V \setminus \{0\}$ given by $x \mapsto x^{-1}$:

$$S^1 = U \coprod_{\phi} V.$$

Let $p = 0 \in U$. Then $\{U, V\}$ is an orbit cover, since the orbits are $p \in U$ and $S^1 \setminus \{p\} = V$. A source-connected integration over U is given by the action groupoid $\mathcal{G}_U = \mathbb{R} \ltimes U$ of $\mathbb{R}$ on U by rescaling, with source and target maps

$$s_U : (t, x) \mapsto x$$
$$t_U : (t, x) \mapsto e^t x.$$

Over V, the algebroid T_pS^1 is simply the tangent bundle, so a source-connected integration is $\mathrm{Pair}(V) = V \times V$. We now glue $\mathcal{G}_U$ to $\mathcal{G}_V$ via the map

$$(t, x) \mapsto (e^{-t}x^{-1}, x^{-1}),$$

an isomorphism of subgroupoids from $\mathbb{R} \ltimes (U \setminus \{0\})$ to $(\mathrm{Pair}(V \setminus 0))^c = \mathrm{Pair}(V_+) \times \mathrm{Pair}(V_-)$, where

$V_\pm = \{y \in \mathbb{R} \mid \pm y > 0\}$. The resulting groupoid is a source-connected integration of T_pS^1, diffeomorphic as a smooth surface to the nontrivial line bundle over S^1.

Combining Theorem 6.1.4 with Theorem 2.1.26, we may express the gluing theorem in terms of explicit choices of normal subgroupoids on the open sets of an orbit cover, as follows.

Let A be a Lie algebroid over M, let $\{U_i\}_{i\in I}$ be an orbit cover for M, and for all $i,j \in I$, let $\widetilde{\mathcal{G}}_i$ and $\widetilde{\mathcal{G}}_{ij}$ be source-simply-connected integrations of A over U_i and $U_i \cap U_j$, respectively. The canonical groupoid morphism $\widetilde{\mathcal{G}}_{ij} \to \widetilde{\mathcal{G}}_i|_{U_i\cap U_j}$ induces a morphism $\mathtt{P}_{ij} : \Lambda(\widetilde{\mathcal{G}}_i) \to \Lambda(\widetilde{\mathcal{G}}_{ij})$ of posets of étale, totally disconnected, normal Lie subgroupoids.

Corollary 6.1.6. *With notation as above, the category of integrations* $\mathtt{Gpd}(A)$ *is equivalent to the fiber product of posets* $\Lambda(\widetilde{\mathcal{G}}_i)$ *over the maps* $\mathtt{P}_{ij}$*, that is:*

$$\mathtt{Gpd}(A) \simeq \lim\left(\prod_{i\in I}\Lambda(\widetilde{\mathcal{G}}_i) \xrightarrow{\mathtt{P}} \prod_{i,j\in I}\Lambda(\widetilde{\mathcal{G}}_{ij})\right).$$

In other words, any integration of A is uniquely specified by the choice of a collection of étale, totally disconnected, normal Lie subgroupoids $\mathcal{N}_i \subset \widetilde{\mathcal{G}}_i$, for all $i \in I$, such that $\mathtt{P}_{ij}(\mathcal{N}_i) = \mathtt{P}_{ji}(\mathcal{N}_j)$ in $\widetilde{\mathcal{G}}_{ij}$.

Proof. By Part *iii)* of Theorem 6.1.4, the restriction of an integrating groupoid to the given orbit cover defines a functor in the forward direction, which is faithful and full by Theorem 2.1.26.

The essential surjectivity is shown as follows. Given an element $\{\mathcal{N}_i\}_{i\in I}$ of the fiber product, we construct an integration $\mathcal{G}$ as in (6.1.1), where $\mathcal{G}_i = \widetilde{\mathcal{G}}_i/\mathcal{N}_i$ and we have groupoid isomorphisms $\phi_{ij} : (\mathcal{G}_i|_{U_{ij}})^c \to (\mathcal{G}_j|_{U_{ij}})^c$ uniquely determined by the condition $\mathtt{P}_{ij}(\mathcal{N}_i) = \mathtt{P}_{ji}(\mathcal{N}_j)$. The non-trivial conditions of Theorem 6.1.4 are that ϕ_{ij} restrict to an isomorphism

$$\phi_{ij} : (\mathcal{G}_i|_{U_{ij}})^c \cap (\mathcal{G}_i|_{U_{ik}})^c \xrightarrow{\cong} (\mathcal{G}_j|_{U_{ij}})^c \cap (\mathcal{G}_j|_{U_{ik}})^c \tag{6.1.2}$$

and $\phi_{ki}\phi_{jk}\phi_{ij} = \mathsf{id}$.

First, we note the general fact that for a source-connected integration $\mathcal{G} \rightrightarrows M$ of A, an orbit $\mathcal{O}$ and an open set $U \subset M$, we have that

$$(\mathcal{G}|_U)^c|_{\mathcal{O}} = (\mathcal{G}|_{U\cap\mathcal{O}})^c.$$

For any $g_i \in (\mathcal{G}_i|_{U_{ij}})^c$, let $\mathcal{O}_x$ be the orbit containing $x = s(g_i) \in M$. By the orbit cover property, we have $\mathcal{O}_x \subset U_l$ for some $l \in I$. Note that $g_i \in (\mathcal{G}_i|_{U_{ij}})^c|_{\mathcal{O}_x} \subset (\mathcal{G}_i|_{U_{ij}\cap U_l})^c = ((\mathcal{G}_i|_{U_{ij}})^c|_{U_l})^c$.

We first show the cocycle condition $\phi_{li}\phi_{jl}\phi_{ij}(g_i) = g_i$ holds when l is involved. To see this, we note that by applying the restriction functor $\mathtt{R}_{U_{ij}\cap U_l}$ to ϕ_{il} and ϕ_{jl}, we obtain isomorphisms

$$\phi_{il} : (\mathcal{G}_i|_{U_{ij}\cap U_l})^c \xrightarrow{\cong} (\mathcal{G}_l|_{U_{ij}})^c, \qquad \phi_{jl} : (\mathcal{G}_j|_{U_{ij}\cap U_l})^c \xrightarrow{\cong} (\mathcal{G}_l|_{U_{ij}})^c.$$

This implies that $\phi_{li}\phi_{jl}\phi_{ij}(g_i)$ is defined. Moreover, $\mathtt{Lie}(\phi_{li}\phi_{jl}\phi_{ij})$ is the identity map on $\mathtt{Lie}(\mathcal{G}_i)|_{U_{ij}\cap U_l}$. Since $(\mathcal{G}_i|_{U_{ij}\cap U_l})^c$ is source-connected, it follows that $\phi_{li}\phi_{jl}\phi_{ij}$ must be the identity map on $(\mathcal{G}_i|_{U_{ij}\cap U_l})^c$, and we conclude that $\phi_{li}\phi_{jl}\phi_{ij}(g_i) = g_i$.

For general $k \in I$, let us now assume $g_i \in (\mathcal{G}_i|_{U_{ij}})^c \cap (\mathcal{G}_i|_{U_{ik}})^c$. We write $g_j = \phi_{ij}(g_i)$, $g_k = \phi_{ik}(g_i)$ and $g_l = \phi_{il}(g_i)$. Since $\phi_{lj}\phi_{kl}(g_k) = \phi_{lj}(g_l) = g_j$, we conclude that $g_j \in (\mathcal{G}_j|_{U_{jk}})^c$, establishing (6.1.2).

The general cocycle condition follows from

$$\phi_{ki}\phi_{jk}\phi_{ij}(g_i) = \phi_{li}\phi_{kl}\phi_{lk}\phi_{jl}\phi_{lj}\phi_{il}(g_i) = g_i.$$

By Part *i)* of Theorem 6.1.4, the coproduct $\mathcal{G} = \coprod_{i\in I} \mathcal{G}_i / \sim$ is a Lie groupoid over M integrating A. Applying the restriction functor, we obtain $(\mathcal{G}|_{U_i})^c = \mathcal{G}_i$, so we recover the normal subgroupoids $\mathcal{N}_i = \ker(\widetilde{\mathcal{G}}_i \to \mathcal{G}_i)$, as required by essential surjectivity. □

Proposition 6.1.7. *Let A be an integrable Lie algebroid over M, and let $\{U_i\}_{i\in I}$ be a locally finite orbit cover of M. If for each $i \in I$, the groupoid $\mathcal{G}_i \rightrightarrows U_i$ in Theorem 6.1.4 is Hausdorff, then the coproduct groupoid $\mathcal{G} \rightrightarrows M$ is Hausdorff.*

Sketch of proof. Since M is Hausdorff, the non-trivial case occurs when the source and target maps coincide on distinct points $g, h \in \mathcal{G}$, that is, $s(g) = s(h)$ and $t(g) = t(h)$. Since $\mathcal{G}$ is source-connected, there exists an orbit $\mathcal{O}$ of A containing both $s(g)$ and $t(g)$. By the orbit cover property, $\mathcal{O}$ is contained in some U_i, and so g, h have representatives in $\mathcal{G}_i$, which is Hausdorff by assumption.

□

6.2 Examples: log tangent case and log symplectic case

In this section we make use of Theorem 6.1.4 to classify integrations of the log tangent algebroid $T(M, D)$ associated to a closed hypersurface $D \subset M$ as well as the lie algebroid $T^*_\pi M$ of a log symplectic structure.

6.2.1 Choosing an orbit cover

We choose an orbit cover for the pair (M, D) as follows: V is the complement of the closed hypersurface D, and U is a tubular neighbourhood of D, chosen so that the tubular neighbourhoods of different connected components of D do not intersect.

We index connected components as follows: let $\mathtt{D} = \pi_0(D)$ and $\mathtt{V} = \pi_0(V)$, so that

$$V = \coprod_{i\in\mathtt{V}} V_i, \qquad U = \coprod_{j\in\mathtt{D}} U_j.$$

It is convenient to partition $\mathtt{D}$ into two subsets, $\mathtt{D} = \mathtt{E} \coprod \mathtt{H}$, where $\mathtt{E}, \mathtt{H}$ are the sets of connected components of D with orientable and non-orientable normal bundles, respectively. It is also convenient, following [34], to represent this information as a graph, as follows.

Definition 6.2.1. With notation as above, the *graph of* (M, D) is the following graph with half-edges (a half-edge is an edge with only one end attached to a vertex).

- The vertices, $\mathtt{V} = \pi_0(V)$, index the components of the complement of D.
- The edges, $\mathtt{E}$, index the components of D with orientable normal bundle; an edge $j \in \mathtt{E}$ joins the pair of vertices representing the open components on either side of D_j (note that these may coincide, in which case the edge becomes a loop).
- The half-edges, $\mathtt{H}$, index the components of D with non-orientable normal bundle; a half-edge $j \in \mathtt{H}$ is attached to a vertex $i \in \mathtt{V}$ if $D_j \subset \overline{V_i}$.

Example 6.2.2. Example 4.1.13 concerns a hypersurface $D \subset \mathbb{R}P^2$ with two connected components. We choose an orbit cover consisting of the complement $V = \mathbb{R}P^2 \setminus D$, with two connected components, and a tubular neighbourhood U of D, with two connected components.

The corresponding graph, shown above, has two vertices, one edge, and one half-edge, as D has two connected components, one of which has nontrivial normal bundle.

The orbit cover described above has the property that the Lie algebroid A (which is either $T(M, D)$ or $T^*_\pi M$) restricts to the tangent algebroid on V and $U \cap V$, i.e. $A|_V = TV$ and $A|_{U\cap V} = T(U \cap V)$. Since the orbit cover has two open sets, Corollary 6.1.6 implies that the category of integrations of A can be described as the following fiber product.

$$\begin{array}{ccc} \mathtt{Gpd}(A) & \longrightarrow & \mathtt{Gpd}(A|_U) \\ \downarrow & & \downarrow \mathtt{P}_U \\ \mathtt{Gpd}(TV) & \xrightarrow[\mathtt{P}_V]{} & \mathtt{Gpd}(T(U \cap V)) \end{array} \tag{6.2.1}$$

The fundamental groupoids of V and $U \cap V$ provide two of the source-simply-connected integrations required to apply Corollary 6.1.6. As described in Example 2.1.27, we may further simplify the bottom row of (6.2.1) by restricting $\Pi_1(V)$ and $\Pi_1(U \cap V)$ to a set of basepoints for the underlying spaces, described in the next section, §6.2.2. This will render $\mathtt{P}_V$ into a morphism between posets of normal subgroups of the fundamental groups $\pi_1(V), \pi_1(U \cap V)$. The choice of basepoints will also be convenient for the description of $\mathtt{P}_U$ in §6.2.4 and §6.2.6.

6.2.2 Choosing basepoints

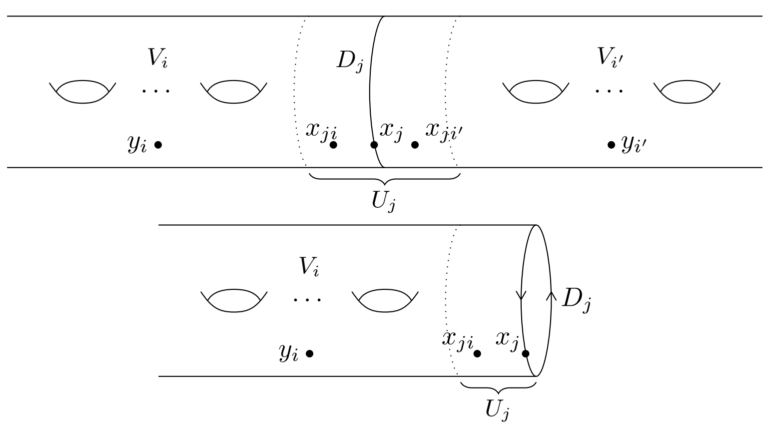


Figure 6.1: Choice of basepoints in the cases $j \in \mathtt{E}$ (above) and $j \in \mathtt{H}$ (below).

We make the following choice of basepoints, as illustrated in Figure 6.1:

- For each $i \in \mathtt{V}$, choose $y_i \in V_i$.
- For each $j \in \mathtt{D}$, choose $x_j \in D_j$.

– For each $j \in \mathtt{E}$, choose basepoints $x_{ji}, x_{ji'}$ in $V_i \cap U$ and $V_{i'} \cap U$ respectively, where $V_i, V_{i'}$ are the open components on either side of D_j. Choose these in such a way that they are sent to x_j by a neighbourhood retraction $r_j : U_j \to D_j$.

– For each $j \in \mathtt{H}$, choose a basepoint $x_{ji} \in V_i \cap U$, where V_i is the open component surrounding D_j. Choose it so that it is sent to x_j by a neighbourhood retraction $r_j : U_j \to D_j$.

Once basepoints are chosen as above, we obtain a simplification of the bottom row of the fiber product diagram (6.2.1). Namely, we obtain equivalences

$$\mathtt{Gpd}(TV) \simeq \prod_{i \in \mathtt{V}} \Lambda(\pi_1(V_i, y_i)),$$
$$\mathtt{Gpd}(T(U \cap V)) \simeq \prod_{i \in \mathtt{V}, j \in \mathtt{D}} \Lambda(\pi_1(V_i \cap U_j, x_{ji})).$$

With respect to this decomposition, the restriction functor $\mathtt{P}_V$ has the following simple description, by an argument as in Example 2.1.27.

Proposition 6.2.3. *The restriction functor $\mathtt{P}_V$ taking integrations of TV to integrations of $T(U \cap V)$ may be described as a poset map from normal subgroups of $\pi_1(V_i, y_i)$ to normal subgroups of $\pi_1(U_j \cap V_i, x_{ji})$: it is the pullback by the group homomorphism*

$$\delta_* : \pi_1(U_j \cap V_i, x_{ji}) \to \pi_1(V_i, y_i), \qquad \gamma \mapsto \delta\gamma\delta^{-1}$$

induced by the choice of a path δ from y_i to x_{ji} in V_i. This map on normal subgroups $N \mapsto \delta_*^{-1}(N)$ is independent of the choice of δ.

To obtain a complete description of $\mathtt{Gpd}(A)$, all that remains is to describe $\mathtt{Gpd}(A_U)$ and $\mathtt{P}_U$ in (6.2.1). Since U is the disjoint union of tubular neighbourhoods U_j of components D_j, $j \in \mathtt{D}$, the problem reduces to a local investigation: we need only describe the source-simply-connected groupoid integrating A_{U_j} and its poset of étale, totally disconnected normal Lie subgroupoids.

6.2.3 Local normal form: log tangent case

Let D be a connected manifold and $p : N \to D$ a real line bundle. We may describe the source-simply-connected integration $\mathcal{G}$ of $T(\mathrm{tot}(N), D)$ as follows. The restriction of $T(\mathrm{tot}(N), D)$ to $D \subset \mathrm{tot}(N)$ is the Atiyah algebroid $\mathtt{At}(N)$ of the bundle N, which has source-simply-connected integration given by the holonomy groupoid $\mathcal{H}ol(N) \rightrightarrows D$, defined by

$$\mathcal{H}ol(N) = \{(\gamma, a) \mid \gamma \in \Pi_1 D,\ a : N_{s_0(\gamma)} \xrightarrow{\cong} N_{t_0(\gamma)}\}.$$

Moreover, $\mathcal{H}ol(N)$ acts on N and the action groupoid

$$\mathcal{G} = \mathcal{H}ol^c(N) \ltimes N \rightrightarrows \mathrm{tot}(N) \tag{6.2.2}$$

is the ssc integration of $T(N, D)$. We divide the subsequent argument into two halves, as N may be orientable or not. We also use N to denote $\mathrm{tot}(N)$, when convenient.

Proposition 6.2.4. *Let $p : N \to D$ be orientable, and let $x \in D \subset N$ be a basepoint. We denote the two connected components of $N \setminus D$ by N^+ and N^-, and choose base points $x^\pm \in N^\pm$ such that $p(x^\pm) = x$. We also define $r = p|_{N \setminus D}$.*

1. *The integrations of $T(N, D)$ are classified by triples*

$$(K^+, K, K^-) \tag{6.2.3}$$

   of normal subgroups $K^+ \subset \pi_1(N^+, x^+)$, $K \subset \pi_1(D, x)$, and $K^- \subset \pi_1(N^-, x^-)$ such that

$$K \subset r_* K^+ \quad \text{and} \quad K \subset r_* K^-.$$

2. *Hausdorff integrations are those triples such that $K = r_* K^+ = r_* K^-$.*

3. *Morphisms between integrations correspond to componentwise inclusion for the associated triple of normal subgroups.*

4. *The fundamental group of the source fiber at x is isomorphic to K.*

5. *Restricting the integration of $T(N, D)$ to $N^\pm$, we obtain the integration of $T(N^\pm)$ given by*

$$\Pi_1(N^\pm)/\mathcal{N}^\pm,$$

   where $\mathcal{N}^\pm$ is the unique totally disconnected normal Lie subgroupoid of $\Pi_1(N^\pm)$ with isotropy $K^\pm$ at $x^\pm$.

Proof. Using a trivialization $\text{tot}(N) \cong \mathbb{R} \times D$ with $N^+ = \mathbb{R}^+ \times D$, $N^- = \mathbb{R}^- \times D$, we express the ssc integration $\mathcal{G} \rightrightarrows N$ as in (6.2.2) explicitly,

$$\mathcal{G} = (\mathbb{R}^+ \ltimes \mathbb{R}) \times \Pi_1(D).$$

By Theorem 2.1.26, the integrations of $T(N, D)$ are classified by the étale, totally disconnected, normal Lie subgroupoids of $\mathcal{G} \rightrightarrows N$, with the Hausdorff integrations requiring the normal subgroupoids be closed.

Let $\mathcal{N}$ be a closed, étale, totally disconnected, normal Lie subgroupoid of $\mathcal{G}$. Since $\mathcal{G}|_{N^+} \cong \Pi_1(N^+)$, its isotropy K^+ at x^+ is a normal subgroup of $\pi_1(N^+, x^+)$. Explicitly, we have $\mathcal{N}|_{N^+} = \{1\} \times \mathbb{R}^+ \times \mathcal{K}^+$ where $\mathcal{K}^+$ is the normal subgroupoid of $\Pi_1 D$ induced by $r_* K^+$. Likewise, we have $\mathcal{N}|_{N^-} = \{1\} \times \mathbb{R}^- \times \mathcal{K}^-$ where $\mathcal{K}^-$ is the normal subgroupoid of $\Pi_1 D$ induced by $r_* K^-$.

We have $\mathcal{N}|_D = \{1\} \times \{0\} \times \mathcal{K}$ where $\mathcal{K}$ is the normal subgroupoid of $\Pi_1 D$ induced by K. Indeed, if $\mathcal{N}$ contains a point $p = (a, 0, \gamma) \in \mathcal{G}$ where $a \neq 1$, then $\mathcal{N}$ contains both $\mathbb{R}^+ \times \{0\} \times \mathsf{id}(D)$ and the identity bisection $\mathsf{id}(N) = \{1\} \times \mathbb{R} \times \mathsf{id}(D)$, the union of which is not a manifold.

To summarize, we have three normal subgroups $r_* K^-$, K and $r_* K^+$ of $\pi_1(D, x)$ inducing three normal subgroupoids $\mathcal{K}^-$, $\mathcal{K}$ and $\mathcal{K}^+$ of $\Pi_1 D$ respectively; and

$$\mathcal{N} = \left(\{1\} \times \mathbb{R}^- \times \mathcal{K}^-\right) \coprod \left(\{1\} \times \{0\} \times \mathcal{K}\right) \coprod \left(\{1\} \times \mathbb{R}^+ \times \mathcal{K}^+\right) \tag{6.2.4}$$

which is a regular submanifold $\mathcal{G}$ if and only if $K \subset r_* K^-$ and $K \subset r_* K^+$, obtaining *i)*. Moreover, $\mathcal{N}$ is a closed submanifold if and only if $K = r_*K^- = r_*K^+$, obtaining *ii)*. The results *iii)*, *iv)* and *v)* follow from the construction. □

Proposition 6.2.5. *Let $p : N \to D$ be non-orientable, and choose base points $x \in D$ and $x' \in N \setminus D$ such that $p(x') = x$. Also, let $r = p|_{N\setminus D}$.*

1. *The integrations of $T(N, D)$ are classified by pairs*

$$(K', K) \tag{6.2.5}$$

of normal subgroups $K' \subset \pi_1(N \setminus D, x')$ and $K \subset \pi_1(D, x)$ such that $K \subset r_*K'$.

2. *Hausdorff integrations are those pairs such that $K = r_*K'$.*

3. *Morphisms between integrations correspond to componentwise inclusion for the associated pair of normal subgroups.*

4. *The fundamental group of the source fiber at x is isomorphic to K.*

5. *Restricting the integration of $T(N, D)$ to $N \setminus D$, we obtain the integration of $T(N \setminus D)$ given by*

$$\Pi_1(N \setminus D)/\mathcal{N}',$$

where $\mathcal{N}'$ is the unique totally disconnected normal Lie subgroupoid of $\Pi_1(N \setminus D)$ with isotropy K' at x'.

Proof. The line bundle N determines a double cover $\varsigma : \widetilde{D} \to D$ and $\widetilde{N} = \varsigma^*N$ is a trivial line bundle, which admits an involution τ such that $\mathrm{tot}(\widetilde{N})/\tau = \mathrm{tot}(N)$. Note that τ induces an involution on the ssc integration $\widetilde{G} \rightrightarrows \widetilde{N}$ of $T(\widetilde{N}, \widetilde{D})$. The ssc integration of $T(N, D)$ is the quotient

$$(\widetilde{\mathcal{G}} \rightrightarrows \widetilde{N})/\tau.$$

The τ-equivariant version of Proposition 6.2.4 yields the desired results.

□

Propositions 6.2.4 and 6.2.5 may be applied to the open sets of an orbit cover of a log symplectic manifold. Using the notation developed in §6.2.1 and §6.2.2 , we state the precise result.

Theorem 6.2.6 (Local classification). *Let D_j, U_j, and V_i be as in §6.2.1 and choose basepoints as in §6.2.2.*

1. *If D_j has orientable normal bundle, then the integrations of $T(U_j, D_j)$ are classified by triples*

$$(K_{ji}, K_j, K_{ji'}) \tag{6.2.6}$$

of normal subgroups $K_{ji} \subset \pi_1(U_j \cap V_i, x_{ji})$, $K_j \subset \pi_1(D_j, x_j)$, and $K_{ji'} \subset \pi_1(U_j \cap V_{i'}, x_{ji'})$, which are compatible with the projection $r : U_j \setminus D_j \to D_j$ of the punctured tubular neighbourhood, in the sense

$$K_j \subset r_*K_{ji} \text{ and } K_j \subset r_*K_{ji'}. \tag{6.2.7}$$

2. *If D_j has non-orientable normal bundle, then the integrations of $T(U_j, D_j)$ are classified by pairs (K_{ji}, K_j) of normal subgroups as above, such that*

$$K_j \subset r_* K_{ji}. \tag{6.2.8}$$

3. *Morphisms between integrations correspond to componentwise inclusion for the associated triple (or pair) of normal subgroups.*

4. *Restricting the integration of $T(U_j, D_j)$ given by (6.2.6) to $U_j \cap V_i$, we obtain the integration of $T(U_j \cap V_i)$ defined by*

$$\Pi_1(U_j \cap V_i)/\mathcal{N}_{ji}, \tag{6.2.9}$$

where $\mathcal{N}_{ji}$ is the unique totally disconnected normal Lie subgroupoid with isotropy K_{ji} at x_{ji}. Similarly, in the orientable case, the restriction to $U_j \cap V_{i'}$ yields $\Pi_1(U_j \cap V_{i'})/\mathcal{N}_{ji'}$, where $\mathcal{N}_{ji'}$ is the subgroupoid with isotropy $K_{ji'}$ at $x_{ji'}$.

5. *The fundamental group of the source fiber over $x_j \in D_j$ is isomorphic to K_j; in particular, the source-simply-connected integration is obtained when all subgroups in the triple (or pair) are trivial.*

6. *Hausdorff integrations are those for which the inclusions (6.2.7), (6.2.8) are equalities.*

Remark 6.2.7. If the normal bundle of D_j is orientable, r_* is an isomorphism, so we may view the groups (6.2.6) as subgroups of the same group $\pi_1(D_j, x_j)$. Consequently, condition (6.2.7) is simply that the normal subgroup K_j must lie in the intersection $K_{ji} \cap K_{ji'}$. For Hausdorff integrations, all three groups must coincide.

In the non-orientable case, r_* is an injection of $\pi_1(U_j \cap V_i, x_{ji})$ onto the kernel of the first Stiefel-Whitney class $w_1 : \pi_1(D_j, x_j) \to \mathbb{Z}/2\mathbb{Z}$ of the normal bundle of D_j. So, we may view K_{ji} as a normal subgroup of $\ker w_1$, and condition (6.2.8) then states that $K_j \subset K_{ji}$. In the Hausdorff case, this is an equality (in particular, this implies K_{ji} is normal in $\pi_1(D_j, x_j)$).

6.2.4 Log tangent integrations

Fix a tubular neighbourhood U_j of a single connected component D_j of the hypersurface D, and choose basepoints x_j, x_{ji} and, if $j \in \mathtt{E}$, $x_{ji'} \in U_j \cap V_{i'}$, as described in §6.2.2.

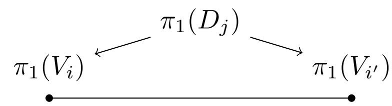
In §6.2.3, we construct the source-simply-connected groupoid $\widetilde{\mathcal{G}}_{U_j}$ integrating $T(U_j, D_j)$, compute its poset of étale, totally disconnected normal Lie subgroupoids (as well as the subposet of closed subgroupoids), and describe the restriction functor $\mathtt{P}_{U_j}$.

Using Theorem 6.2.6, we are able to fill in the diagram (6.2.1) and give a global description of the category of integrations $\mathtt{Gpd}(T(M, D))$. We will phrase the fiber product in terms of the graph introduced in §6.2.1, using the basepoint choices from §6.2.2.

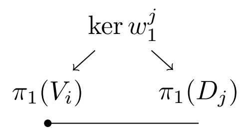
Definition 6.2.8. The graph of groups associated to (M, D) is defined as follows. Let Γ be the graph associated to (M, D) in Definition 6.2.1. Let δ_{ji} be paths joining y_i to x_{ji} for all $i \in \mathtt{V}, j \in \mathtt{D}$. We label Γ with groups and homomorphisms in the following way, using the identifications in Remark 6.2.7.

– To each vertex $i \in \mathtt{V}$, we associate the group $\pi_1(V_i, y_i)$.

– To each edge $j \in \texttt{E}$ joining i to i', we associate the group $\pi_1(D_j, x_j)$, together with the induced homomorphisms $(\delta_{ji})_*, (\delta_{ji'})_*$ from $\pi_1(D_j, x_j)$ to the corresponding vertex groups $\pi_1(V_i, y_i)$ and $\pi_1(V_{i'}, y_{i'})$.



– To each half-edge $j \in \texttt{H}$ attached to i, we associate the inclusion of groups $\ker w_1^j \hookrightarrow \pi_1(D_j, x_j)$ determined by the Stiefel-Whitney class w_1^j of ND_j, together with the induced homomorphism $(\delta_{ji})_* : \ker w_1^j \to \pi_1(V_i, y_i)$.



Theorem 6.2.9 (Global classification)**.** *Given the graph of groups associated to (M, D) in Definition 6.2.8, the category of integrations $\texttt{Gpd}(T(M, D))$ is equivalent to the poset whose elements consist of:*

1. *A normal subgroup K_i of each vertex group $\pi_1(V_i, y_i), i \in \texttt{V}$,*
2. *A normal subgroup K_j for each edge group $\pi_1(D_j, x_j), j \in \texttt{E}$, such that*

$$K_j \subset {\delta_{ji}}_*^{-1}(K_i) \text{ and } K_j \subset {\delta_{ji'}}_*^{-1}(K_{i'}), \tag{6.2.10}$$

   where j joins i to i',

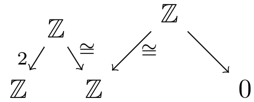
3. *A normal subgroup K_j of each half-edge group $\pi_1(D_j, x_j), j \in \texttt{H}$, such that*

$$K_j \subset {\delta_{ij}}_*^{-1}(K_i) \quad \text{in} \quad \ker w_1^j, \tag{6.2.11}$$

   where j is attached to i.

The partial order is componentwise inclusion for the corresponding normal subgroups, and the fundamental group of the source fiber over any of the basepoints is given by Theorem 6.2.6. In particular, the source-simply-connected integration is obtained when all subgroups over vertices, edges, and half-edges are trivial. Finally, the Hausdorff integrations are those for which the inclusions in 6.2.10 and 6.2.11 are all equalities.

Example 6.2.10. The log tangent integrations for Example 6.2.2 are classified using the following graph of groups:



There is only one nontrivial vertex group, one edge group, and one half-edge group. We choose a subgroup $n\mathbb{Z}$ of the vertex group $\mathbb{Z}$, for some $n = 0, 1, \ldots$, and condition (6.2.10) forces the edge subgroup to be $n'\mathbb{Z} \subset n\mathbb{Z}$. Then on the half-edge, we must choose a subgroup $2n''\mathbb{Z} \subset 2n\mathbb{Z}$. Integrations are therefore in bijection with the poset

$$\{(n, n', n'') \in \mathbb{N}^3 \ : \ n|n' \text{ and } n|n''\} \cup \{(0, 0, 0)\}.$$

The partial order is componentwise divisibility, and $(0, 0, 0)$ is the least element, corresponding to the source-simply-connected integration. For Hausdorff integrations, first we have the condition that the edge group coincides with the pullbacks from the left and right vertices, which are $n\mathbb{Z}$ and $\mathbb{Z}$, respectively. This implies $n = 1$ and $n' = 1$. secondly, the half-edge group must coincide with $2n\mathbb{Z}$, so we have $n'' = 1$. Therefore, we conclude that only one of the integrations is Hausdorff, corresponding to the point $(1, 1, 1)$ in the above set. This is, of course, the log pair groupoid constructed in §5.1.

Let U_j be a tubular neighbourhood of one connected component D_j of the degeneracy locus D of a proper log symplectic manifold (M, π), and choose basepoints x_j, x_{ji} and, if $j \in \mathtt{E}$, $x_{ji'} \in U_j \cap V_{i'}$, as described in §6.2.2.

In §6.2.5, we construct the source-simply-connected groupoid $\widetilde{\mathcal{G}}_{U_j}$ integrating $T^*_\pi U_j$, compute its poset of étale, totally disconnected normal Lie subgroupoids (as well as the subposet of closed subgroupoids), and describe the restriction functor $\mathtt{P}_{U_j}$.

6.2.5 Local normal form: log symplectic case

Following Proposition 4.1.11, the local normal form of a proper log symplectic structure near a connected component of its degeneracy locus is built from the following data:

- A compact, connected, symplectic manifold (F, ω), a symplectomorphism $\varphi : F \to F$, and a constant $\lambda \in \mathbb{R}^+$, which determine the Poisson mapping torus

  $$D = S^1_\lambda \ltimes_\varphi F$$

  as defined in (4.1.4), with projection $f : D \to S^1_\lambda$;

- A line bundle L over D induced by an $\mathbb{Z}$-equivariant line bundle over F with a metric connection ∇;

- An orientable or non-orientable line bundle over S^1_λ which we call Q^+, Q^-, respectively.

Then the total space of $N = f^*Q^\pm \otimes L$ inherits a log symplectic structure π, as explained in Proposition 4.1.9. We construct the ssc symplectic groupoid of $(\mathrm{tot}(N), \pi)$ as an action groupoid of the fiber product of two groupoids.

The first groupoid is the monodromy groupoid, obtained by lifting $\varphi : F \to F$ to $\varphi : \Pi_1 F \to \Pi_1 F$:

$$\mathrm{Mon}(D, f) = S^1_\lambda \ltimes_\varphi \Pi_1 F = \frac{\Pi_1 F \times \mathbb{R}}{(\gamma, t) \sim (\varphi(\gamma), t + \lambda)}.$$

This is a Lie groupoid over D, and using the metric connection ∇ on L, we obtain an action of $\mathrm{Mon}(D, f)$ on L.

In the case that the orientable line bundle Q^+ is chosen, the second groupoid, $\mathcal{A}^+$, is defined to be the trivial bundle of groups $\mathcal{A}^+ = A \times S^1_\lambda$, where $A = \mathbb{R}^+ \ltimes \mathbb{R}$ is the group of affine transformations of the plane. Using a trivialization $Q^+ = S^1_\lambda \times \mathbb{R}$ with coordinates (t, r), we obtain an action of $\mathcal{A}^+$ on Q^+ via

$$\begin{pmatrix} t \\ r \end{pmatrix} \mapsto \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix}, \text{ for } \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \in A. \tag{6.2.12}$$

In the case that Q^- is chosen, we define a groupoid $\mathcal{A}^-$ by taking the quotient of $A \times S^1_{2\lambda}$ by the involution σ defined by

$$A \times S^1_{2\lambda} \xrightarrow{\sigma} A \times S^1_{2\lambda},$$
$$\left(\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}, t \right) \mapsto \left(\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}, t + \lambda \right).$$

Then $\mathcal{A}^- = (A \times S^1_{2\lambda})/\sigma$ is a nontrivial bundle of groups over S^1_λ. By expressing Q^- as the quotient $(t, r) \sim (t + \lambda, -r)$, it inherits an $\mathcal{A}^-$–action as in (6.2.12).

Having the groupoid $\mathrm{Mon}(D, f)$ over D, and pulling back $\mathcal{A}^\pm$ to a groupoid over D, we form the fiber product groupoid

$$\mathcal{H} = f^*\mathcal{A}^\pm \times_D \mathrm{Mon}(D, f),$$

obtaining a Lie groupoid over D which acts on $N = f^*Q^\pm \otimes L$ by combining the action of $\mathrm{Mon}(D, f)$ on L and the action of $\mathcal{A}^\pm$ on $Q^\pm$. Finally, the action groupoid

$$\mathcal{G} = \mathcal{H} \ltimes N \tag{6.2.13}$$

is the source-simply-connected integration of the Poisson algebroid $T^*_\pi(\mathrm{tot}(N))$[1]

Proposition 6.2.11. *Let $p : N \to D$, as defined above, be orientable, let $x \in D$ be a basepoint on the zero section, and let $\iota : F \hookrightarrow D$ be the inclusion of a symplectic leaf through x. We denote the two connected components of $N \setminus D$ by N^+ and N^-, and choose base points $x^\pm \in N^\pm$ such that $p(x^\pm) = x$. Also, we let $r = p|_{N \setminus D}$.*

1. *The Hausdorff integrations of $T^*_\pi N$ are classified by pairs*

$$(K^+, K^-) \tag{6.2.14}$$

   of normal subgroups $K^+ \subset \pi_1(N^+, x^+)$, and $K^- \subset \pi_1(N^-, x^-)$ such that

$$\iota_*^{-1}(r_* K^+) = \iota_*^{-1}(r_* K^-).$$

2. *Morphisms between integrations correspond to componentwise inclusion for the associated pair of normal subgroups.*

3. *The fundamental group of the source fiber at x is isomorphic to $\iota_*^{-1}(r_* K^+)$.*

4. *Restricting the integration of $T^*_\pi N$ to $N^\pm$ and $N^\pm$, we obtain the integration of $T(N^\pm)$*

$$\Pi_1(N^\pm)/\mathcal{N}^\pm$$

[1] This local normal form for the groupoid is implicit in [16, p. 32].

where $\mathcal{N}^{\pm}$ is the unique totally disconnected normal Lie subgroupoid of $\Pi_1(N^{\pm})$ *with isotropy* $K^{\pm}$ *at* $x^{\pm}$.

Proof. Using a trivialization, we may decompose $\mathrm{tot}(N)$ as follows:

$$\begin{aligned}\mathrm{tot}(N) &= (S^1_\lambda \ltimes_\varphi F) \times \mathbb{R}\\ &= \{(t,x,r) \mid t \in S^1_\lambda,\ x \in F,\ r \in \mathbb{R},\ (t,\gamma,r) \sim (t+\lambda, \varphi(\gamma), r)\}.\end{aligned}$$

Similarly, we write the ssc integration $\mathcal{G} \rightrightarrows N$ defined in (6.2.13) explicitly:

$$\begin{aligned}\mathcal{G} &= (\mathcal{A}^+ \times_D (S^1_\lambda \ltimes_\varphi \Pi_1 F)) \ltimes N\\ &= \{(a,b,t,\gamma,r) \mid a \in \mathbb{R}^+,\ b \in \mathbb{R},\ t \in S^1_\lambda,\ \gamma \in \Pi_1 F,\ r \in \mathbb{R},\\ &\qquad (a,b,t,\gamma,r) \sim (a,b,t+\lambda,\varphi(\gamma),r)\},\end{aligned} \tag{6.2.15}$$

where the source and target maps to $\mathrm{tot}(N)$ are given by

$$s : (a,b,t,\gamma,r) \mapsto (t, s_0(\gamma), r), \quad t : (a,b,t,\gamma,r) \mapsto (t+br, t_0(\gamma), ar).$$

By Theorem 2.1.26, it suffices to classify closed, étale, totally disconnected, normal Lie subgroupoids of $\mathcal{G} \rightrightarrows \mathrm{tot}(N)$. So, let $\mathcal{N}$ be such a subgroupoid.

First note that the isotropy groups of the groupoid $\mathcal{G}$ are given as follows: at $x^+ = (t,x,r) \in N \setminus D$, we have $r > 0$ and the isotropy group of $\mathcal{G}$ is given by

$$\mathcal{G}(x',x') = \big((\{1\} \times \tfrac{\lambda}{r}\mathbb{Z} \times \{t\}\big) \ltimes \pi_1(F,x)\big) \times \{r\}, \tag{6.2.16}$$

while at $x = (t,x,0) \in D$, the isotropy group of $\mathcal{G}$ is given by

$$\mathcal{G}(x,x) = \big((\mathbb{R}^+ \ltimes \mathbb{R}) \times \{t\}\big) \times \pi_1(F,x)\big) \times \{0\}. \tag{6.2.17}$$

For a point $p = (a,b,t,\gamma,0) \in \mathcal{G}|_D$ such that $a \neq 1$ or $b \neq 0$, if we take a small neighbourhood U_p around p, then $U_p \cap \mathcal{N}|_{N\setminus D} = \varnothing$. Since $\dim \mathcal{N} = \dim M$, it follows that $\dim(U_p \cap \mathcal{N}|_D) = \dim M$. However since $s : \mathcal{N}|_D \to D$ is a submersion, we must have that $\dim\big(\mathcal{N} \cap s^{-1}(s(p))\big) = 1$, which is an contradiction.

Therefore the subgroupoid $\mathcal{N}|_D$ must take the form

$$\mathcal{N}|_D = \big((\{1\} \times \{0\} \times S^1_\lambda) \ltimes \mathcal{H}\big) \times \{0\}.$$

where $\mathcal{H}$ is the normal subgroupoid $\Pi_1 F$ induced by a normal subgroup $H \subset \pi_1(F,x)$. We denote the isotropy group of $\mathcal{N}$ at x^+ by K^+. By (6.2.16), K^+ is a normal subgroup of $\frac{\lambda}{r}\mathbb{Z} \ltimes \pi_1(F,x)$. If we define $H^+ = K^+ \cap \pi_1(F,x) = \iota_*^{-1}(r_* K^+)$, which is a φ-invariant normal subgroup of $\pi_1(F,x)$, then as we take the limit $r \to 0$, the condition that $\mathcal{N}$ is closed implies $H^+ = H$. Similarly, we have $H^- = H$.

Since $H^+ = H = H^-$, we conclude that

$$\iota_*^{-1}(r_* K^+) = \iota_*^{-1}(r_* K^-),$$

obtaining *i)*. The results *ii)*, *iii)* and *iv)* follow from the construction. □

Using the same strategy as in Proposition 6.2.5, we address the non-orientable case:

Proposition 6.2.12. *Let $p : N \to D$, as defined above, be non-orientable, let $x \in D$ be a basepoint on the zero section, and let $\iota : F \hookrightarrow D$ be the inclusion of a symplectic leaf through x. Choose a base point $x' \in N \setminus D$, and let $r = p|_{N\setminus D}$.*

1. *The Hausdorff integrations of $T^*_\pi N$ are classified by a normal subgroup $K' \subset \pi_1(N \setminus D, x')$.*
2. *Morphisms between integrations correspond to inclusions of associated normal subgroups.*
3. *The fundamental group of the source fiber at x is isomorphic to $\iota_*^{-1}(r_* K')$.*
4. *Restricting the integration of $T^*_\pi N$ to $N \setminus D$, we obtain the integration of $T(N \setminus D)$ given by*

$$\Pi_1(N \setminus D)/\mathcal{N}'$$

where $\mathcal{N}'$ is the unique closed, totally disconnected, normal Lie subgroupoid of $\Pi_1(N \setminus D)$ with isotropy K' at x'.

Propositions 6.2.11 and 6.2.12 may be applied to the open sets of an orbit cover of a log symplectic manifold. Using the notation developed in §6.2.1 and §6.2.2, we state the precise result.

Theorem 6.2.13 (Local classification)**.** *Let D_j, U_j, and V_i be as in §6.2.1, where the hypersurface D is the degeneracy locus of a proper symplectic manifold (M, π), and choose basepoints as in §6.2.2.*

1. *If ND_j is orientable, then the Hausdorff integrations of $T^*_\pi U_j$ are classified by pairs*
$$(K_{ji}, K_{ji'}) \tag{6.2.18}$$
of normal subgroups $K_{ji} \subset \pi_1(U_j \cap V_i, x_{ji})$ and $K_{ji'} \subset \pi_1(U_j \cap V_{i'}, x_{ji'})$, which are compatible with the projection $r : U_j \setminus D_j \to D_j$ of the punctured tubular neighbourhood and inclusion map $\iota_j : F_j \to D_j$, in the sense
$$(\iota_j)_*^{-1}(r_* K_{ji}) = (\iota_j)_*^{-1}(r_* K_{ji'}), \tag{6.2.19}$$
as subgroups of $\pi_1(F_j, x_j)$.
2. *If ND_j is non-orientable, then the Hausdorff integrations of $T^*_\pi U_j$ are classified by a normal subgroup K_{ji} as above, with no additional constraint.*
3. *Morphisms between integrations correspond to componentwise inclusion for the associated pair of normal subgroups.*
4. *The restriction of an integration of $T^*_\pi U_j$ given by (6.2.18) to $U_j \cap V_i$ is an integration of TU_j, obtained in the same way as in Equation 6.2.9.*
5. *The fundamental group of the source fiber over the point $x_j \in D_j$ is isomorphic to $(\iota_j)_*^{-1}(r_* K_{ji})$.*

Remark 6.2.14. For D_j orientable, r_* is an isomorphism, so we may view the groups (6.2.18) as subgroups of the same group $\pi_1(D_j, x_j)$. Condition (6.2.19) is simply that their preimages in $\pi_1(F_j, x_j)$ agree.

6.2.6 Hausdorff log symplectic integrations

Let U_j be a tubular neighbourhood of one connected component D_j of the degeneracy locus D of a proper log symplectic manifold (M, π), and choose basepoints x_j, x_{ji} and, if $j \in \mathtt{E}$, $x_{ji'} \in U_j \cap V_{i'}$, as described in §6.2.2.

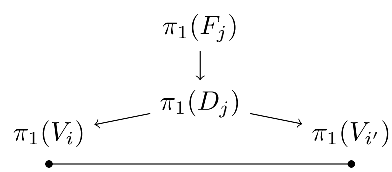
In §6.2.5, we construct the source-simply-connected groupoid $\widetilde{\mathcal{G}}_{U_j}$ integrating $T^*_\pi U_j$, compute its poset of étale, totally disconnected normal Lie subgroupoids (as well as the subposet of closed subgroupoids), and describe the restriction functor $\mathtt{P}_{U_j}$.

Theorem 6.2.13 and Equation 6.2.1 allow us to give an explicit description of the category of Hausdorff integrations $\mathtt{Gpd}^{\mathcal{H}}(T^*_\pi M)$. We will express the coproduct in terms of the graph introduced in §6.2.1, using the basepoint choices from §6.2.2.

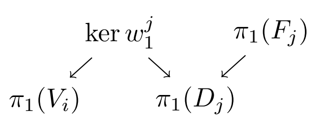
Definition 6.2.15. The graph of groups associated to a proper log symplectic manifold (M, π) is defined as follows. Let Γ be the graph associated to (M, D) in Definition 6.2.1. Let δ_{ji} be paths joining y_i to x_{ji} for all $i \in \mathtt{V}, j \in \mathtt{D}$, and let $\iota_j : F_j \to D_j$ be the inclusion of the symplectic leaf through x_j. We label the graph with groups and homomorphisms in the following way, using the identifications in Remark 6.2.14.

- To each vertex $i \in \mathtt{V}$, we associate the group $\pi_1(V_i, y_i)$.

- To each edge $j \in \mathtt{E}$ joining i to i', we associate the morphism of groups $(\iota_j)_* : \pi_1(F_j, x_j) \to \pi_1(D_j, x_j)$, together with the induced homomorphisms $(\delta_{ji})_*, (\delta_{ji'})_*$ from $\pi_1(D_j, x_j)$ to the vertex groups $\pi_1(V_i, y_i)$ and $\pi_1(V_{i'}, y_{i'})$, as below:



- To each half-edge $j \in \mathtt{H}$ attached to $i \in \mathtt{V}$, we associate the inclusion of groups $\ker w_1^j \hookrightarrow \pi_1(D_j, x_j)$ determined by the Stiefel-Whitney class w_1^j of ND_j, as well as the morphism $(\iota_j)_* : \pi_1(F_j, x_j) \to \pi_1(D_j, x_j)$, and finally the induced homomorphism $(\delta_{ji})_* : \ker w_1^j \to \pi_1(V_i, y_i)$.



Remark 6.2.16. For a proper log symplectic manifold (M, π), the adjoint integration of $T^*_\pi M$ is a symplectic groupoid by Remark 5.2.4. It follows from Remark 2.2.12 that all other integrations of $T^*_\pi M$ are also symplectic groupoids.

Theorem 6.2.17 (Global classification)**.** *Given the graph of groups from Definition 6.2.15, the category of Hausdorff symplectic groupoids* $\mathtt{Gpd}^{\mathcal{H}}(T^*_\pi M)$ *for a proper log symplectic manifold* (M, π) *is equivalent to the poset whose elements consist of a family of normal subgroups* K_i *of each vertex group* $\pi_1(V_i, y_i), i \in \mathtt{V}$, *such that if* i, i' *share an edge* $j \in \mathtt{E}$, *then* $K_i, K_{i'}$ *coincide upon restriction to* $\pi_1(F_j, x_j)$, *that is,*

$$(\iota_j)_*^{-1}(\delta_{ji})_*^{-1} K_i = (\iota_j)_*^{-1}(\delta_{ji})_*^{-1} K_{i'}.$$

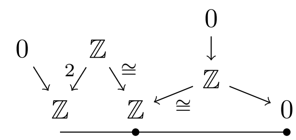
The partial order is componentwise inclusion for corresponding normal subgroups, and the fundamental group of the source fiber over $x_j, j \in D$, is given by the restriction

$$(\iota_j)_*^{-1}(\delta_{ji})_*^{-1}K_i, \tag{6.2.20}$$

for i attached to $j \in D$.

Corollary 6.2.18. *The source-simply-connected integration of a proper log symplectic manifold is Hausdorff if and only if, for each symplectic leaf F contained in the degeneracy hypersurface D, and for each class $\gamma \in \pi_1(F)$ on which the first Stiefel-Whitney class of ND vanishes, the push-off of γ is nonzero in the fundamental group of the adjacent open symplectic leaf or pair of leaves.*

Example 6.2.19. The Hausdorff symplectic groupoids of the Poisson structure described in Example 4.1.13 are classified using the following graph of groups:



For any choice of subgroup $n\mathbb{Z} \subset \mathbb{Z}$ of the only nontrivial vertex group, the conditions of Theorem 6.2.17 are trivially satisfied. Hence, the integrations are classified by the poset $\mathbb{N} \cup \{0\}$, where the partial order is divisibility and 0 is the minimum. Applying (6.2.20) to the diagram above, we see that the source fiber over a point on the degeneracy locus has trivial fundamental group for any choice of n. Hence 0 represents the source-simply-connected integration.

A similar argument may be used to construct the Hausdorff source-simply-connected integration of any log symplectic 2-manifold, whose existence was shown in [27].

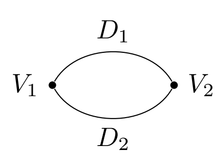
Example 6.2.20. Let (M, π) be the log symplectic 4-manifold constructed as a $\mathbb{Z}^2$ quotient of $\mathbb{R}^2 \times T^2$, equipped with the Poisson structure

$$\left(\sin(2\pi y)\tfrac{\partial}{\partial x} \wedge \tfrac{\partial}{\partial y}\right) \oplus \omega^{-1},$$

where ω is the standard symplectic form on T^2. The action is given by

$$(n_1, n_2) : (x, y, p) \mapsto (x + n_1, y + n_2, \varphi^{n_2}(p)),$$

for some fixed $\varphi \in \mathrm{SL}(2, \mathbb{Z})$. The degeneracy locus of (M, π) is the union of two mapping tori D_1, D_2, each isomorphic to $S^1 \ltimes_\varphi T^2$. The open symplectic leaves V_1, V_2 are each homotopic to $S^1 \ltimes_\varphi T^2$. All of these have fundamental group $\mathbb{Z} \ltimes_{\varphi_*} \mathbb{Z}^2$. A symplectic leaf $F_j \subset D_j$ is isomorphic to (T^2, ω) and its fundamental group $\pi_1(F_j, *) = \mathbb{Z}^2$ is a normal subgroup of $\mathbb{Z} \ltimes_{\varphi_*} \mathbb{Z}^2$.



The graph of (M, π) is shown above. By Theorem 6.2.17, each of its Hausdorff symplectic groupoids is given by a pair of normal subgroups N_1, N_2 of $\mathbb{Z} \ltimes_{\phi_*} \mathbb{Z}^2$ such that $N_1 \cap (\mathbb{Z} \times \mathbb{Z}) = N_2 \cap (\mathbb{Z} \times \mathbb{Z})$. In particular, taking both N_1, N_2 to be trivial, (6.2.20) yields a trivial fundamental group for the source fibers over D_1, D_2, so that we obtain a Hausdorff source-simply-connected symplectic groupoid.

6.3 Examples: divisors on a Riemann surface

In this section, we make use of Theorem 6.1.4 to classify integrations of the log tangent algebroid $T\mathsf{X}(-\mathsf{D})$ where X is a Riemann surface and D is a divisor on X.

Let $\mathsf{D} = d_1p_1 + d_2p_2 + \ldots + d_kp_k$. We choose an orbit cover for (X, D) as follows: V is the complement of $\{p_1, \ldots, p_k\}$, and U is a disk neighbourhood of $\{p_1, \ldots, p_k\}$, chosen so that for $i \neq j$, the disk neighbourhoods of p_i and p_j do not intersect. Let us choose a basepoint $y \in V$. For each $j = 1, 2, \ldots, k$, we write the disk neighbourhood of p_j as U_j and choose a basepoint $x_j \in U_j \setminus p_j$. Furthermore, we choose a path δ_j from y to x_j.

As in §6.2, the orbit cover described above has the property that the holomorphic Lie algebroid $T\mathsf{X}(-\mathsf{D})$ restricts to the tangent algebroid on V and $U \cap V$, i.e. $T\mathsf{X}(-\mathsf{D})|_V = TV$ and $T\mathsf{X}(-\mathsf{D})|_{U\cap V} = T(U \cap V)$. As before, Corollary 6.1.6 implies that the category of integrations of A can be described as the following fiber product.

$$\begin{array}{ccc} \mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})) & \longrightarrow & \mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_U) \\ \big\downarrow & & \big\downarrow \mathtt{P}_U \\ \mathtt{Gpd}(TV) & \xrightarrow[\mathtt{P}_V]{} & \mathtt{Gpd}(T(U\cap V)) \end{array} \tag{6.3.1}$$

The fundamental groupoids $\Pi_1(V)$ and $\Pi_1(U \cap V)$ provide two of the source-simply-connected integrations required to apply Corollary 6.1.6. As before, we have

$$\mathtt{Gpd}(TV) \simeq \mathtt{\Lambda}(\pi_1(V, y)),$$
$$\mathtt{Gpd}(T(U \cap V)) \simeq \prod_{j=1}^{k} \mathtt{\Lambda}\left(\pi_1(U_j \setminus p_j, x_j)\right).$$

As before, choosing paths δ_j renders $\mathtt{P}_V$ into a morphism between posets of normal subgroups of the fundamental groups $\pi_1(V)$ and $\pi_1(U \cap V)$, i.e. the pullback by the group homomorphism

$$(\delta_j)_* : \pi_1(U_j \setminus p_j, x_j) \to \pi_1(V, y), \qquad \gamma \mapsto \delta_j \gamma \delta_j^{-1}.$$

Notice that $U_j \setminus p_j$ is a punctured disk, and $\pi_1(U_j \setminus p_j, x_j) \simeq \mathbb{Z}$. Consequently for each $j = 1, 2, \ldots, k$, a choice of a normal subgroup $N \subset \pi_1(V)$ induces a subgroup $n_j\mathbb{Z}$ of $\mathbb{Z}$, i.e. N induces a non-negative integer n_j.

It remains to describe $\mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_U)$ and $\mathtt{P}_U$. For this, we describe a local normal form of the ssc integration of $T\mathbb{C}(-d \cdot 0)$, which we denote by $\Pi_1(\mathbb{C}, d \cdot 0)$, and its posets of étale, totally disconnected, normal Lie subgroupoids.

When $d = 1$, the groupoid $\Pi_1(\mathbb{C}, -0)$ is the action groupoid $\mathbb{C} \ltimes \mathbb{C}$ associated to the exponential action. Explicitly, we have

$$\begin{aligned} s(\lambda, z) &= z \\ t(\lambda, z) &= \exp(\lambda) z \\ m\left((\lambda_2, z_2), (\lambda_1, z_1)\right) &= (\lambda_2 + \lambda_1, z_1). \end{aligned}$$

Proposition 6.3.1. *1. The integrations of $T\mathbb{C}(-0)$ are classified by $(\mathbb{N} \times \mathbb{N}) \coprod \{(1,0)\}$.*

2. Hausdorff integrations are the ones given by $m = 1$, i.e. these are classified by the non-negative integers, $\mathbb{Z}_{\geq 0}$.

Proof. By Theorem 2.1.26, the integrations of $T\mathbb{C}(-0)$ are classified by the étale, totally disconnected, normal Lie subgroupoids of $\Pi_1(\mathbb{C}, -0) = \mathbb{C} \ltimes \mathbb{C}$, with the Hausdorff integrations requiring the normal subgroupoids be closed.

Away from $0 \in \mathbb{C}$, we have $\mathbb{C} \ltimes \mathbb{C}^* \cong \Pi_1 \mathbb{C}^*$. The étale, totally disconnected, normal Lie subgroupoid of $\mathbb{C} \ltimes \mathbb{C}^*$ are of the form $2\pi k\mathrm{i}(n\mathbb{Z}) \ltimes \mathbb{C}^*$ where n is a non-negative integer.

If we require the étale, totally disconnected, normal Lie subgroupoid to be closed in $\mathbb{C} \ltimes \mathbb{C}$, then it must be of the form $2\pi k\mathrm{i}(n\mathbb{Z}) \ltimes \mathbb{C}$. If we do not require it to be closed, then it is of the form

$$(2\pi k\mathrm{i}(mn\mathbb{Z}) \ltimes \{0\}) \coprod (2\pi k\mathrm{i}(n\mathbb{Z}) \ltimes \mathbb{C}^*) \subset \mathbb{C} \ltimes \mathbb{C}$$

where m is a positive integer. Note that when $n = 0$, different values of m will yield the same result.

Note that the case $m = 1$ corresponds to the closed étale, totally disconnected, normal subgroupoid. □

For $d > 1$, the groupoid $\Pi_1(\mathbb{C}, d \cdot 0)$ may be described as the $(d-1)$-fold groupoid blowup of $\Pi_1(\mathbb{C}, 0)$ along $\mathsf{id}(0)$. Explicitly, $\Pi_1(\mathbb{C}, d \cdot 0)$ may be identified with $\mathbb{C} \times \mathbb{C}$, with groupoid structure

$$\begin{aligned} s(\lambda, z) &= z \\ t(\lambda, z) &= \exp(\lambda z^{d-1}) z \\ m\left((\lambda_2, z_2), (\lambda_1, z_1)\right) &= (\lambda_2 \exp((d-1)\lambda_1 z_1^{d-1}) + \lambda_1, z_1). \end{aligned}$$

Proposition 6.3.2. *For $d > 1$, all integrations of $T\mathbb{C}(-d \cdot 0)$ are Hausdorff, and they are classified by the non-negative integers $\mathbb{Z}_{\geq 0}$.*

Proof. By Theorem 2.1.26, the integrations of $T\mathbb{C}(-d \cdot 0)$ are classified by the étale, totally disconnected, normal Lie subgroupoids of $\Pi_1(\mathbb{C}, d \cdot 0)$. These are of the form

$$\{(\lambda, z) \in \Pi_1(\mathbb{C}, d \cdot 0) \cong \mathbb{C} \times \mathbb{C} \mid \lambda z^{d-1} = 2\pi k\mathrm{i}n, \quad k \in \mathbb{Z}\}$$

where n is a non-negative integer. Incidentally, all these subgroupoids are closed inside $\Pi_1(\mathbb{C}, d \cdot 0)$. □

To summarize, we have the following results:

1. If $d_j = 1$, then

$$\begin{aligned} \mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_{U_j}) &\simeq (\mathbb{N} \times \mathbb{N}) \coprod \{(1,0)\}, \\ \mathtt{Gpd}^{\mathcal{H}}(T\mathsf{X}(-\mathsf{D})|_{U_j}) = \mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_{U_j \setminus p_j}) &= \mathtt{Gpd}^{\mathcal{H}}(T\mathsf{X}(-\mathsf{D})|_{U_j \setminus p_j}) \simeq \mathbb{Z}_{\geq 0}. \end{aligned} \tag{6.3.2}$$

2. If $d_j > 1$, then

$$\mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_{U_j}) = \mathtt{Gpd}^{\mathcal{H}}(T\mathsf{X}(-\mathsf{D})|_{U_j}) = \mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_{U_j\setminus p_j}) = \mathtt{Gpd}^{\mathcal{H}}(T\mathsf{X}(-\mathsf{D})|_{U_j\setminus p_j}) \simeq \mathbb{Z}_{\geq 0}. \tag{6.3.3}$$

3.

$$\begin{aligned} \mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_U) &= \prod_{j=1}^k \mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_{U_j}) \simeq \left(\prod_{d_j=1} \mathbb{N}\times\mathbb{Z}_{\geq 0}\right) \times \left(\prod_{d_j>1} \mathbb{Z}_{\geq 0}\right), \\ \mathtt{Gpd}^{\mathcal{H}}(T\mathsf{X}(-\mathsf{D})|_U) &= \prod_{j=1}^k \mathtt{Gpd}^{\mathcal{H}}(T\mathsf{X}(-\mathsf{D})|_{U_j}) \simeq \prod_{j=1}^k \mathbb{Z}_{\geq 0}, \\ \mathtt{Gpd}(T\mathsf{X}(-\mathsf{D})|_{U\cap V}) &= \mathtt{Gpd}^{\mathcal{H}}(T\mathsf{X}(-\mathsf{D})|_{U\cap V}) \simeq \prod_{j=1}^k \mathbb{Z}_{\geq 0}. \end{aligned} \tag{6.3.4}$$

4. $\mathtt{P}_U$ is the projection onto the $\mathbb{Z}_{\geq 0}$ factor.

Theorem 6.3.3 (Global classification)**.** *Let X be a Riemann surface, and let $\mathsf{D} = d_1p_1+d_2p_2+\ldots+d_kp_k$ be a divisor on X. The category of integrations $\mathtt{Gpd}(T\mathsf{X}(-\mathsf{D}))$ is equivalent to the product of the following posets:*

1. *The poset of normal subgroups of the fundamental group of the complement of the divisor, i.e. $\Lambda(\pi_1(V,y))$.*

2. *A choice of $N \in \Lambda(\pi_1(V,y))$ induces a normal subgroup of $\pi_1(U_j \setminus p_j, x_j) \simeq \mathbb{Z}$, which defines a non-negative integer n_j. For n_j and $d_j = 1$, we have an additional choice of a positive integer m_j.*

   The category of Hausdorff integrations $\mathtt{Gpd}^{\mathcal{H}}(T\mathsf{X}(-\mathsf{D}))$ is equivalent to $\Lambda(\pi_1(V,y))$.

Bibliography

[1] Camilo Arias Abad and Marius Crainic. The Weil algebra and the Van Est isomorphism. *Ann. Inst. Fourier (Grenoble)*, 61(3):927–970, 2011.

[2] Gregory Arone and Marja Kankaanrinta. On the functoriality of the blow-up construction. *Bull. Belg. Math. Soc. Simon Stevin*, 17(5):821–832, 2010.

[3] H. Bursztyn and O. Radko. Gauge equivalence of Dirac structures and symplectic groupoids. *Ann. Inst. Fourier (Grenoble)*, 53(1):309–337, 2003.

[4] Alberto S. Cattaneo and Giovanni Felder. Poisson sigma models and symplectic groupoids. In *Quantization of singular symplectic quotients*, volume 198 of *Progr. Math.*, pages 61–93. Birkhäuser, Basel, 2001.

[5] A. Coste, P. Dazord, and A. Weinstein. Groupoïdes symplectiques. In *Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2*, volume 87 of *Publ. Dép. Math. Nouvelle Sér. A*, pages i–ii, 1–62. Univ. Claude-Bernard, Lyon, 1987.

[6] Marius Crainic and Rui Loja Fernandes. Integrability of Lie brackets. *Ann. of Math. (2)*, 157(2):575–620, 2003.

[7] Marius Crainic and Rui Loja Fernandes. Integrability of Poisson brackets. *J. Differential Geom.*, 66(1):71–137, 2004.

[8] Claire Debord. Holonomy groupoids of singular foliations. *J. Differential Geom.*, 58(3):467–500, 2001.

[9] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin, 1970.

[10] Jean-Paul Dufour and Nguyen Tien Zung. *Poisson structures and their normal forms*, volume 242 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2005.

[11] Ryushi Goto. Rozansky-Witten invariants of log symplectic manifolds. In *Integrable systems, topology, and physics (Tokyo, 2000)*, volume 309 of *Contemp. Math.*, pages 69–84. Amer. Math. Soc., Providence, RI, 2002.

[12] Marco Gualtieri and Songhao Li. Symplectic groupoids of log symplectic manifolds. *to appear in IMRN*, 2013.

[13] Marco Gualtieri, Songhao Li, and Brent Pym. Stokes groupoids. 2013.

[14] Marco Gualtieri and Brent Pym. Poisson modules and degeneracy loci. *to appear in Proceedings of London Mathematical Society*, 2013.

[15] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Codimension one symplectic foliations and regular Poisson structures. *Bull. Braz. Math. Soc. (N.S.)*, 42(4):607–623, 2011.

[16] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Symplectic and poisson geometry of b–manifolds. 2012.

[17] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[18] Eli Hawkins. A groupoid approach to quantization. *J. Symplectic Geom.*, 6(1):61–125, 2008.

[19] Philip J. Higgins and Kirill C. H. Mackenzie. Fibrations and quotients of differentiable groupoids. *J. London Math. Soc. (2)*, 42(1):101–110, 1990.

[20] Dale Husemoller. *Elliptic curves*, volume 111 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2004. With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen.

[21] M. V. Karasëv. Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50(3):508–538, 638, 1986.

[22] André Lichnerowicz. Les variétés de Poisson et leurs algèbres de Lie associées. *J. Differential Geometry*, 12(2):253–300, 1977.

[23] Jiang-Hua Lu and Alan Weinstein. Groupoïdes symplectiques doubles des groupes de Lie-Poisson. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(18):951–954, 1989.

[24] Jiang-Hua Lu and Alan Weinstein. Poisson Lie groups, dressing transformations, and Bruhat decompositions. *J. Differential Geom.*, 31(2):501–526, 1990.

[25] Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[26] Kirill C. H. Mackenzie and Ping Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.

[27] David Martínez Torres. A note on the separability of canonical integrations of Lie algebroids. *Math. Res. Lett.*, 17(1):69–75, 2010.

[28] Rafe Mazzeo and Richard B. Melrose. Pseudodifferential operators on manifolds with fibred boundaries. *Asian J. Math.*, 2(4):833–866, 1998.

[29] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.

[30] Bertrand Monthubert. Pseudodifferential calculus on manifolds with corners and groupoids. *Proc. Amer. Math. Soc.*, 127(10):2871–2881, 1999.

[31] Victor Nistor. Groupoids and the integration of Lie algebroids. *J. Math. Soc. Japan*, 52(4):847–868, 2000.

[32] A. Polishchuk. Algebraic geometry of Poisson brackets. *J. Math. Sci. (New York)*, 84(5):1413–1444, 1997. Algebraic geometry, 7.

[33] Jean Pradines. Théorie de Lie pour les groupoïdes différentiables. Calcul différenetiel dans la catégorie des groupoïdes infinitésimaux. *C. R. Acad. Sci. Paris Sér. A-B*, 264:A245–A248, 1967.

[34] Olga Radko. A classification of topologically stable Poisson structures on a compact oriented surface. *J. Symplectic Geom.*, 1(3):523–542, 2002.

[35] J. A. Schouten. On the differential operators of first order in tensor calculus. In *Convegno Internazionale di Geometria Differenziale, Italia, 1953*, pages 1–7. Edizioni Cremonese, Roma, 1954.

[36] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition.

[37] Izu Vaisman. *Lectures on the geometry of Poisson manifolds*, volume 118 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994.

[38] Alan Weinstein. Symplectic groupoids and Poisson manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 16(1):101–104, 1987.

[39] Alan Weinstein. Coisotropic calculus and Poisson groupoids. *J. Math. Soc. Japan*, 40(4):705–727, 1988.

[40] Alan Weinstein. Blowing up realizations of Heisenberg-Poisson manifolds. *Bull. Sci. Math.*, 113(4):381–406, 1989.

[41] Alan Weinstein. Groupoids: unifying internal and external symmetry. A tour through some examples. *Notices Amer. Math. Soc.*, 43(7):744–752, 1996.

[42] Alan Weinstein. The modular automorphism group of a Poisson manifold. *J. Geom. Phys.*, 23(3-4):379–394, 1997.

[43] Alan Weinstein. Groupoids: unifying internal and external symmetry. A tour through some examples. In *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, volume 282 of *Contemp. Math.*, pages 1–19. Amer. Math. Soc., Providence, RI, 2001.

[44] Michael Lennox Wong. *Hecke Modifications, Wonderful Compactifications and Isomonodromy*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–McGill University (Canada).