Construction of Lie groupoids

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Outline

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  Poisson groupoids

The blow-up construction
  Blowing up Lie groupoids
  Blowing up Poisson groupoids

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  The gluing construction

Application to log symplectic manifolds
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Lie groupoids

A Lie groupoid $\mathcal{G}$ over the base manifold $M$ is a category such that

- the set of objects is $M$, and the set of arrows $\mathcal{G}$ is a manifold;
- the arrows are invertible;
- the source $s$ and target $t$ are submersions;
- the multiplication $m$ and the identity $id$ are smooth.
Introduction: Lie theory  Lie groupoids

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- the multiplication $m$ and the identity $\text{id}$ are smooth.

The structure maps are summarized by the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{G}_t \times_s \mathcal{G} & \xrightarrow{m} & \mathcal{G} \\
\downarrow i \quad \quad \quad \quad \quad \quad \downarrow t \quad \quad \quad \quad \quad \quad \quad \downarrow \text{id} \\
M & \xleftarrow{\text{id}} & \mathcal{G}_t \times_s \mathcal{G}
\end{array}
$$
Lie algebroids

A Lie algebroid $A$ over the base manifold $M$ is a vector bundle $A \to M$ with a Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$ and an anchor map $a : A \to TM$ that preserves the bracket and satisfies the Leibniz rule

$$[X, fY] = f[X, Y] + a(X)(f)Y. \quad (1.1)$$
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Lie functor

For a Lie algebroid $\mathcal{G} \rightrightarrows M$, the vector bundle

$$\text{Lie}(\mathcal{G}) = \text{id}^* \ker (Ts : T\mathcal{G} \to TM) \quad (1.2)$$

with the bracket on left invariant vector fields, and the anchor $Tt : \text{Lie}(\mathcal{G}) \to TM$, is Lie algebroid.
Poisson groupoids

Poisson manifolds

A Poisson manifold is a manifold $M$ with a bivector $\pi \in \mathfrak{x}^2(M)$ such that

$$[\pi, \pi] = 0,$$

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where $[\cdot, \cdot]$ is the Schouten bracket.
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Poisson groupoids
A Poisson groupoid is a Lie groupoid $G \rightrightarrows M$ with a Poisson structure $\sigma$ on $G$ such that the graph of multiplication

$$\text{Graph}(m) = \{(g, h, m(g, h)) \mid (g, h) \in G_t \times sG\}$$  \hspace{1cm} (1.4)

is coisotropic with respect to $\sigma \oplus \sigma \oplus -\sigma$. 
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is coisotropic with respect to $\sigma \oplus \sigma \oplus -\sigma$.

- The pushforward $\pi = s_\ast(\sigma) = -t_\ast(\sigma)$ is Poisson on $M$. 

\[\text{Li (Toronto)}\]
Lie bialgebroids

Poisson algebroids

For a Poisson manifold \((M, \pi)\), the Poisson algebroid \(T^*_\pi M\) is \((T^* M, \pi^\#)\) with the Koszul bracket

\[
[\alpha, \beta] = L_{\pi^\#(\alpha)} \beta - L_{\pi^\#(\beta)} \alpha - d\pi(\alpha, \beta).
\]  

\(1.5\)
A Poisson manifold is a smooth manifold equipped with a Lie bracket on its space of smooth functions that satisfies certain properties. The Koszul bracket on the Poisson algebroid of a Poisson manifold is given by:

\[[\alpha, \beta] = L_{\pi^\#} (\alpha) \beta - L_{\pi^\#} (\beta) \alpha - d\pi (\alpha, \beta)\].

For a Poisson manifold and a coisotropic submanifold \( C \subset M \), the conormal bundle \( N^\ast C \) is a Lie subalgebroid of the Poisson algebroid \( T^\ast \pi M \).
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For a Poisson manifold \((M, \pi)\) and a coisotropic submanifold \(C \subset M\), the conormal bundle \(N^* C\) is a Lie subalgebroid of the Poisson algebroid \(T^*_\pi M\).

Lie functor for Poisson groupoids
For a Poisson groupoid \((G, \sigma) \rightrightarrows (M, \pi)\), the Lie algebroids \(A \doteq \text{Lie}(G)\) and \(A^* \doteq N^* (\text{id}(M))\) form a Lie bialgebroid, and we write

\[
\text{Lie}(G, \sigma) = (A, A^*).
\]

(1.6)
Symplectic groupoids

A symplectic groupoid is a Poisson groupoid $(\mathcal{G}, \sigma) \rightrightarrows (M, \pi)$ such that $\sigma$ is non-degenerate.
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\text{Lie}(\mathcal{G}) = T^*_\pi M, \\
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- Conversely, the ssc groupoid integrating \(T^*_\pi M\) is naturally a symplectic groupoid.
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- Conversely, the ssc groupoid integrating \(T^*_\pi M\) is naturally a symplectic groupoid.

- We will construct the symplectic groupoids of a class of Poisson manifolds, called log symplectic manifolds.
Example: A Log Symplectic Structure on $S^2$

Consider the 2-sphere $S^2$ with the longitude $\theta \in \mathbb{R}$ and latitude $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, with the identification at the poles

$$(\pm \frac{\pi}{2}, \theta) \sim (\pm \frac{\pi}{2}, \theta')$$
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We define the Poisson structure

$$\pi \equiv \left(\frac{1}{\varphi + \frac{\pi}{2}} + \frac{1}{\varphi - \frac{\pi}{2}}\right) \frac{\partial}{\partial \varphi} \wedge \frac{\partial}{\partial \theta}.$$
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$$\pi = \left( \frac{1}{\varphi + \frac{\pi}{2}} + \frac{1}{\varphi - \frac{\pi}{2}} \right) \frac{\partial}{\partial \varphi} \wedge \frac{\partial}{\partial \theta}.$$ 

Note that $\pi$ is nondegenerate away from the equator $\{\varphi = 0\}$, and the degeneracy is linear

$$\lim_{\varphi \to 0} \frac{1}{\varphi} \left( \frac{1}{\varphi + \frac{\pi}{2}} + \frac{1}{\varphi - \frac{\pi}{2}} \right) = -\frac{8}{\pi^2}.$$
Lower elementary modifications

Lower elementary modification of vector bundles

Let $L$ be a closed hypersurface of $M$. Let $A \to M$ be a vector bundle, and $B \to L$ a subbundle of $A|_L$. The lower elementary modification $[A:B]$ of $A$ along $B$ is the vector bundle with sheaf of sections given by

$$[A:B](U) = \{ X \in \Gamma(U, A) \mid X|_L \in \Gamma(U \cap L, B) \}$$

for open sets $U \subset M$. (2.1)
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Lower elementary modification of Lie algebroids
If $A$ is a Lie algebroid, and $B$ is a Lie subalgebroid, then $[A:B]$ is a Lie algebroid.
The blow-up construction Blowing up Lie groupoids

Blowing up Lie groupoids

Theorem (Gualtieri-Li)

Let $\mathcal{G} \xrightarrow{\pi} M$ be a Lie groupoid, and let $\mathcal{H} \xrightarrow{\iota} L$ be a Lie subgroupoid over a closed hypersurface $L$. Let

$$s^{-1}(L) \subset \text{Bl}_\mathcal{H}(\mathcal{G}), \quad t^{-1}(L) \subset \text{Bl}_\mathcal{H}(\mathcal{G})$$

be the proper transforms of $s^{-1}(L)$ and $t^{-1}(L)$.

There is a unique Lie groupoid structure on

$$[\mathcal{G} : \mathcal{H}] = \text{Bl}_\mathcal{H}(\mathcal{G}) \setminus (s^{-1}(L) \cup t^{-1}(L))$$

(2.2)

such that the blow-down map $p : [\mathcal{G} : \mathcal{H}] \to \mathcal{G}$ is groupoid morphism.

Furthermore, we have

$$\text{Lie}([\mathcal{G} : \mathcal{H}]) = [\text{Lie}(\mathcal{G}) : \text{Lie}(\mathcal{H})].$$

(2.3)
Upper elementary modifications

Upper elementary modification of vector bundles
Let $A \to M$ be a vector bundle, and let $B \to L$ be a vector bundle over a closed hypersurface $L$ with a surjective bundle morphism $\phi$

$$K \hookrightarrow A|_L \twoheadrightarrow B. \quad (2.4)$$

Let $f$ be a defining function of $L$, i.e. $f|_L = 0$ and $df|_L \neq 0$. The upper elementary modification $\{A:B\}$ of $A$ along $B$ is the vector bundle with sheaf of sections given by

$$\Gamma(\{A:B\}) = \{ X \in \Gamma(A) \otimes \mathcal{I}^{-1}_L \mid fX|_L \in \Gamma(K) \}. \quad (2.5)$$
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Upper elementary modification of Lie algebroids
If $\phi : A|_L \twoheadrightarrow B$ is a surjective Lie algebroid comorphism, then the upper modification $\{A:B\}$ is a Lie algebroid.
Blowing up Poisson groupoids

Theorem (Li)

Let $(\mathcal{G}, \sigma) \rightrightarrows (M, \pi)$ be a Poisson groupoid, and let $\mathcal{H} \rightrightarrows L$ be a Poisson subgroupoid over a closed hypersurface $L$ such that

$$\text{Lie} (\mathcal{G}, \sigma) = (A, A^*), \quad \text{Lie} (\mathcal{H}, \sigma_{\mathcal{H}}) = (B, B^*).$$

If the induced transverse Poisson structure on $N^*_{\text{id}(x)} \mathcal{H}$ is degenerate for every $x \in M$, then there is a unique multiplicative Poisson structure $\sigma'$ on the blow-up groupoid $[\mathcal{G} : \mathcal{H}] \rightrightarrows M$ such that

$$\text{Lie} ([\mathcal{G} : \mathcal{H}], \sigma') = ([A : B], \{ A^* : B^* \}), \quad (2.6)$$

and the blow-down map $p : [\mathcal{G} : \mathcal{H}] \to \mathcal{G}$ is a Poisson groupoid morphism.
Orbit cover

Orbit cover of a Lie groupoid

An orbit cover of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a locally finite cover $\{U_i\}_{i \in I}$ of $M$ such that each orbit of $\mathcal{G} \rightrightarrows M$ is contained in $U_i$ for some $i \in I$. 
**Orbit cover**

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If $\mathcal{G} \rightrightarrows M$ is source-connected, then $\{U_i\}_{i \in I}$ is also an orbit cover for the underlying Lie algebroid $\text{Lie}(\mathcal{G})$. 
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Restriction of a Lie groupoid
The restriction of a Lie groupoid $\mathcal{G} \rightrightarrows M$ to an open set $U \subset M$, denoted by $(\mathcal{G}|_U)^c$, is the source-connected part of $s^{-1}(U) \cap t^{-1}(U)$. 

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Idea: inspired by [Nistor]
An orbit cover $\{U_i\}_{i \in I}$ enables us to glue Lie groupoids over the open sets $U_i$’s such that the restriction to $U_{ij}$ agree.
The gluing theorem

Theorem (Gualtieri-Li)

For an integrable Lie algebroid $A$ with an orbit cover $\{U_i\}_{i \in I}$, let $G_i \rightrightarrows U_i$ be a source-connected Lie groupoid and let $\phi_{ij} : (G_i|_{U_{ij}})^c \rightarrow (G_j|_{U_{ij}})^c$ be groupoid isomorphisms satisfying $\text{Lie}(\phi_{ij}) = \text{id}$, $\phi_{ii} = \text{id}$, $\phi_{ij} = \phi_{ji}^{-1}$ and the cocycle condition. The fibered coproduct of manifolds

$$G \doteq \coprod_{i \in I} G_i \simeq$$

is a source-connected Lie groupoid integrating $A$, such that $(G|_{U_i})^c = G_i$. Moreover, every source-connected groupoid is obtained in this way.
Illustration: the gluing construction

\[(G|_U)^c \quad \text{and} \quad (G|_V)^c\]
Log symplectic manifolds

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Properness

A log symplectic manifold $(M, \pi)$ is proper if each connected component $D_j$ of the degeneracy locus $D$ is compact and contains a compact symplectic leaf.
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Theorem (Guillemin-Miranda-Pires)

For a proper log symplectic manifold, each $D_j$ is a symplectic mapping torus. In particular, $f_j : D_j \to \gamma_j$ is a symplectic fibre bundle.
Illustration: proper log symplectic manifolds
Symplectic pair groupoid

Idea
The blow-up of the pair groupoid \((\text{Pair}(M), \pi \oplus -\pi)\) along the Poisson subgroupoid \(\text{Pair}_f(D) \cong \bigsqcup_j (D_j \times \gamma_j D_j)\), yields a symplectic groupoid \((\text{Pair}_\pi(M), \sigma)\).
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Theorem (Gualtieri-Li)
For a proper log symplectic manifold $(M, \pi)$, the symplectic pair groupoid $(\text{Pair}_\pi(M), \sigma)$ is the adjoint symplectic groupoid.
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Theorem (Gualtieri-Li)
For a proper log symplectic manifold \((M, \pi)\), the symplectic pair groupoid \((\text{Pair}_\pi(M), \sigma)\) is the adjoint symplectic groupoid.

Corollary (Gualtieri-Li)
For a proper log symplectic manifold \((M, \pi)\), every integration of the Poisson algebroid \(T^*_\pi M\) is a symplectic groupoid.
Illustration: symplectic pair groupoid

Pairc(D) \ s^{-1}(D) = D \times M

id(M)

t^{-1}(D) = M \times D

Pair(M)

Pairf(D)

Pair_\pi(M)

p

Pair(M)

Pair_\pi(D)
Classification of symplectic groupoids

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Theorem (Gualtieri-Li)
For a proper log symplectic manifold \((M, \pi)\), the Hausdorff symplectic groupoids are classified by a family of normal subgroups \(K_i \triangleleft \pi_1(V_i, y_i)\) for each connected component \(V_i \subset (M \setminus D)\) that ‘agree’ when pulling back to the symplectic leaves of the degeneracy locus \(D\).
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The gluing theorem enables us to classify the symplectic groupoids of a proper log symplectic manifold.

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Remark
The precise statement uses a graph with half-edges labelled with groups, which we illustrate by the following example.
Graph

For the log symplectic surface below

\[ \text{Diagram of log symplectic manifold} \]
Graph

For the log symplectic surface below

we associate the graph below:
Graph of groups

In addition, we label the vertices and (half-)edges with the fundamental groups of $V_i$, $D_j$ and the symplectic leaf of $D_j$, and the kernel of the first Stiefel-Whitney class of $ND_j$ with the induced morphisms, as illustrated below:
Graph of groups

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Since the symplectic leaves on the degeneracy locus are points, the symplectic groupoids are classified by a family of normal subgroups for each of $\mathbb{Z}$, $F_2 = \langle a, b \rangle$ and $\mathbb{Z}$. 
Thank you!