Mechanics on fibered manifolds

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Outline

**Lagrangian mechanics**
- Lagrangian mechanics on $TQ$
- Lagrangian mechanics on $A$
- Reduction by groupoid action

**Hamilton-Pontryagin mechanics**
- H-P mechanics on $TQ \oplus T^* Q$
- H-P mechanics on $(A, A^*)$
- Reduction by groupoid action
Q is a manifold.

\( L \in C^\infty(TQ) \) is a smooth function.

\( \mathcal{P}(Q) = \{ q : I \rightarrow Q \mid q \text{ is } C^2 \} \).

\( \delta q \in T_q \mathcal{P}(Q) \) with \( \delta q(0) = \delta q(1) = 0 \).

\( S : \mathcal{P}(Q) \rightarrow \mathbb{R}, \quad S(q) = \int_0^1 L(q(t), \dot{q}(t)) \, dt \).
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**Lagrangian mechanics on** _TQ_

A path \( q \in \mathcal{P}(Q) \) satisfies Hamilton’s variational principle if

\[
\delta S = dS(\delta q) = 0, \quad \forall \delta q.
\]  

(1)

In coordinates, we have the Euler-Lagrange equation:

\[
\frac{\partial L}{\partial q^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}.
\]  

(2)
Lagrangian mechanics

1. $Q$ is a manifold
2. $\rho : A \to TQ$ is a Lie algebroid.
3. $P_{\rho}(A) = \{ a : I \to A \mid \rho(a) = \dot{q} \}$ is the space of $A$-paths, where $q$ is the base path of $a$. 
Q is a manifold

\[ \rho : A \to TQ \] is a Lie algebroid.

\[ \mathcal{P}_\rho(A) = \{ a : I \to A \mid \rho(a) = \dot{q} \} \] is the space of A-paths, where \( q \) is the base path of \( a \).

Choosing \( TQ \)-connection on \( A \), \( \nabla : \Gamma(TQ) \times \Gamma(A) \to \Gamma(A) \), we have two induced \( A \)-connections on \( A \), \( \nabla \) and \( \overline{\nabla} \):

\[ \nabla_X Y = \nabla_{\rho(X)} Y, \quad \overline{\nabla}_X Y = \nabla_{\rho(Y)} X + [X, Y]. \]  \hspace{1cm} (3)
\( Q \) is a manifold

\( \rho : A \to TQ \) is a Lie algebroid.

\( P_\rho(A) = \{ a : I \to A \mid \rho(a) = \dot{q} \} \) is the space of \( A \)-paths, where \( q \) is the base path of \( a \).

Choosing \( TQ \)-connection on \( A \), \( \nabla : \Gamma(TQ) \times \Gamma(A) \to \Gamma(A) \), we have two induced \( A \)-connections on \( A \), \( \nabla \) and \( \overline{\nabla} \):

\[
\nabla X Y = \nabla_{\rho(X)} Y, \quad \overline{\nabla} X Y = \nabla_{\rho(Y)} X + [X, Y].
\]

(3)

An \( A \)-variation of \( a \) is \( \delta a = X_{b,a} \in T_a P_\rho(A) \) where \( b \in P(A) \) is a path in \( A \) such that \( b(0) = 0 \) and \( b(1) = 0 \).

Relative to a chosen \( TM \)-connection \( \nabla \) on \( A \), the horizontal component of \( X_{b,a} \) is \( \rho(b) \), and the vertical component is \( \overline{\nabla} a b \), which are independent of the choice of \( \nabla \).
- $Q$ is a manifold.
- $\rho : A \to TQ$ is a Lie algebroid.
- the space of $A$-paths:
  \[ \mathcal{P}_\rho(A) = \{ a : I \to A \mid \rho(a) = \dot{q}, \ q \text{ is the base path of } a \}. \]
- $\delta a = X_{b,a} \in T_a\mathcal{P}_\rho(A)$ with $b(0) = 0$ and $b(1) = 0$.
- $S : \mathcal{P}_\rho(A) \to \mathbb{R}$, \[ S(a) = \int_0^1 L(a(t)) \, dt. \]
Lagrangian mechanics on $A$ (Weinstein ’96, Martinez ’01, L-S-T)

A $A$-path $a \in \mathcal{P}_\rho(A)$ satisfies Hamilton’s variational principle if

$$\delta S = dS(X_{b,a}) = 0, \quad \text{for all } A\text{-variations } X_{b,a}. \quad (4)$$

Using $\nabla$, we have the Euler-Lagrange-Poincare equation:

$$\rho^*dL^{\text{hor}}(a) + \nabla_a^*dL^{\text{ver}}(a) = 0. \quad (5)$$
- $G \rightrightarrows M$ is a Lie groupoid.
- $\mu : Q \to M$ is a fibered manifold equipped with a free and proper action by $G \rightrightarrows M$.
- $VQ$ is the vertical tangent bundle.
- $L \in \mathcal{C}^\infty (VQ)^G$ is $G$-invariant.
Lagrangian mechanics  Reduction by groupoid action

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1. $VQ/G$ is a Lie algebroid over $Q/G$.
2. $L \in C^\infty (VQ)^G$ reduces to $\ell \in C^\infty (VQ/G)$. 
Lagrangian mechanics Reduction by groupoid action

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The mechanics for \( L \) on \( VQ \) is equivalent to the mechanics for \( \ell \) on \( VQ/G \). That is,

\[
\tilde{a} \in \mathcal{P}_V(Q) \text{ satisfies the E-L-P equation for } L \iff a \in \mathcal{P}_\rho(VQ/G) \text{ satisfies the E-L-P equation for } \ell
\]
Example

1. $G$ is a Lie group.
2. $L \in C^\infty(TQ)^G$ is (left) invariant.
Example

- $G$ is a Lie group.
- $L \in C^\infty(TG)^G$ is (left) invariant.

It follows that $L$ reduces to $\ell \in C^\infty(g)$. The Euler-Lagrange equation for $L$ reduces to the Euler-Poincare equation:

$$\text{ad}_{\xi}^* \mu = \dot{\mu},$$

where $\xi(t) = a(t)$ and $\mu = \frac{\delta \ell}{\delta \xi}$. 
Q is a manifold.

\( L \in C^\infty(TQ) \) is a smooth function.

\( \mathcal{P}(TQ \oplus T^*Q) = \{(q, v, p) : I \to TQ \oplus T^*Q\} \).

\( (\delta q, \delta v, \delta p) \in T_{q,v,p} \mathcal{P}(TQ \oplus T^*Q) \) with \( \delta q(0) = \delta q(1) = 0 \).

\( S : \mathcal{P}(TQ \oplus T^*Q) \to \mathbb{R} \) is defined by

\[
S(q, v, p) = \int_0^1 L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \, dt. \tag{7}
\]
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**H-P mechanics on** \( TQ \oplus T^*Q \)

A path \((q, v, p) \in \mathcal{P}(TQ \oplus T^*Q)\) satisfies H-P principle if

\[
\delta S = dS(\delta q, \delta v, \delta p) = 0, \quad \forall (\delta q, \delta v, \delta p).
\] (8)

In coordinates, we have the implicit E-L equations:

\[
\dot{p}_i = \frac{\partial L}{\partial q^i}, \quad \rho_i = \frac{\partial L}{\partial v^i}, \quad v_i = \dot{q}^i.
\] (9)
\( \rho : A \to TQ \) is a Lie algebroid.
\( L \in C^\infty(A) \) is a smooth function.
\(\rho : A \to TQ\) is a Lie algebroid.

\(L \in C^\infty(A)\) is a smooth function.

\((A, A^*)\)-paths

An \((A, A^*)\)-path consists of

1. an \(A\)-path \(a \in P_\rho(A)\) over the base path \(q \in P(Q)\);
2. a path on \(A\), \(\nu \in P(A)\), over \(q\);
3. a path on \(A^*\), \(\rho \in P(A^*)\), over \(q\);

The space of \((A, A^*)\)-paths: \(P(A, A^*)\).
- $\rho: A \to TQ$ is a Lie algebroid.
- $L \in C^\infty(A)$ is a smooth function.

**(A, A*)-paths**

An (A, A*)-path consists of

1. an A-path $a \in \mathcal{P}_\rho(A)$ over the base path $q \in \mathcal{P}(Q)$;
2. a path on $A$, $v \in \mathcal{P}(A)$, over $q$;
3. a path on $A^*$, $p \in \mathcal{P}(A^*)$, over $q$;

The space of (A, A*)-paths: $\mathcal{P}(A, A^*)$.

**The action $S$**

$S: \mathcal{P}(A, A^*) \to \mathbb{R}$ is defined by

$$S(a, v, p) = \int_0^1 L(v(t)) + \langle p(t), a(t) - v(t) \rangle \, dt.$$ 

(10)
Variations of \((A, A^*)\)-paths

For a \((A, A^*)\)-path \((a, v, p)\),

1. we vary the \(A\)-path \(a\) by an \(A\)-variation \(\delta a = X_{b,a} \in T_a\mathcal{P}_\rho(A)\) with \(b(0) = 0\) and \(b(1) = 0\), whose base variation is \(\delta q \in T_q\mathcal{P}(Q)\);
2. we vary the \(v \in \mathcal{P}(A)\) by a free variation \(\delta v \in T_v\mathcal{P}(A)\) over \(\delta q\);
3. we vary the \(p \in \mathcal{P}(A^*)\) by a free variation \(\delta p \in T_p\mathcal{P}(A^*)\) over \(\delta q\);
Variations of \((A, A^*)\)-paths

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1. we vary the \(A\)-path \(a\) by an \(A\)-variation \(\delta a = X_{b,a} \in T_a\mathcal{P}_\rho(A)\) with \(b(0) = 0\) and \(b(1) = 0\), whose base variation is \(\delta q \in T_q\mathcal{P}(Q)\);
2. we vary the \(v \in \mathcal{P}(A)\) by a free variation \(\delta v \in T_v\mathcal{P}(A)\) over \(\delta q\);
3. we vary the \(p \in \mathcal{P}(A^*)\) by a free variation \(\delta p \in T_p\mathcal{P}(A^*)\) over \(\delta q\);

A \((A, A^*)\)-path \((a, v, p)\) satisfies H-P principle if

\[
\delta S = dS(\delta a, \delta v, \delta p) = 0, \quad \text{for all variations } (\delta a = X_{b,a}, \delta v, \delta p). \quad (11)
\]
Theorem (Li-Stern-Tang)

Choosing a $TM$-connection $\nabla$ on $A$, an $(A, A^*)$-path $(a, v, p)$ satisfies H-P principle if and only if $(a, v, p)$ satisfies the implicit E-L-P equations:

$$\rho^* dL^\text{hor}(v) + \bar{\nabla}^*_a p = 0,$$

$$dL^\text{ver}(v) - p = 0,$$

$$a = v.$$  \hfill (12)
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**Theorem (Li-Stern-Tang)**

The H-P mechanics for $L$ on $VQ \oplus V^*Q$ is equivalent to the H-P mechanics for $\ell$ on $(A, A^*)$ where $A = VQ/G$. That is,

$$(\tilde{q}, \tilde{v}, \tilde{p}) \in \mathcal{P}_V(VQ \oplus V^*Q)$$

satisfies the implicit E-L-P equation for $L$. \iff

$$(a, v, p) \in \mathcal{P}(A, A^*)$$

satisfies the implicit E-L-P equation for $\ell$. 


References


Thank you!