I. True/False

1. The binary operation on a group is completely determined by the identity element and inverse operation. **False**

2. If $\phi : G \to G'$ is a group homomorphism and H is a subgroup of G, then $\phi(H)$ is a subgroup of G'. **True**

3. An inner automorphism of an abelian group must be the identity map. **True**

4. If H and K are subgroups of G such that $H \cap K = \{e\}$ and $G = HK$, then H is a normal subgroup of G. **False**

Counter example: $G = S_3$, $H = \langle (23) \rangle$ and $K = \langle (123) \rangle$.

5. Among all abelian groups of order p^n for some prime number p, the one with the minimum number of subgroups, is the cyclic one. **True**

II. Short Answers

1. In each of the following cases, indicate whether the given subgroup H of S_4 is normal. If H is normal, state what the quotient group S_4/H is.

a) $H = \{e\}$

$\{e\}$ is normal. $S_4 \,/\, \{e\} \cong S_4$.

b) $H = \langle (12) \rangle$

$\langle (12) \rangle$ is not normal.

c) $H = S_3$

S_3 is not normal.

d) $H = A_4$

A_4 is normal. $S_4 \,/\, A_4 \cong \mathbb{Z}/2\mathbb{Z}$.

e) $H = \langle (1234), (14)(23) \rangle$

H is not normal.

2. In each of the following cases, indicate the order of the element g in the given group G.

a) $g = m$, $G = \mathbb{Z}/n\mathbb{Z}$.

The order of m is $\frac{n}{\gcd(m,n)}$.

b) $g = (52743)$, $G = S_8$.

The order of (52743) is 5.

c) $g = 5 + \langle 4 \rangle$, $G = (\mathbb{Z}/12\mathbb{Z}) \,/\, \langle 4 \rangle$.

The order of $5 + \langle 4 \rangle$ is 4.

d) $g = (2, 1) + \langle (4, 4) \rangle$, $G = (\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}) \,/\, \langle (4, 4) \rangle$.

The order of $(2, 1) + \langle (4, 4) \rangle$ is 4.

e) $g = (12345)(5678)(89)$, $G = S_9$.

The order of $(12345)(5678)(89) = (123456789)$ is 9.

III. Proofs

1. Let C be a normal subgroup of A, and let D be a normal subgroup of B. Prove that

   a) $(C \times D) \triangleleft (A \times B)$;

   b) $(A \times B) / (C \times D) \cong (A / C) \times (B / D)$.

Proof.

a) Since $C \triangleleft A$ and $D \triangleleft B$, for all $c \in C$ and $a \in A$, we have $aca^{-1} \in C$; and for all $d \in D$ and $b \in B$, we have $bdb^{-1} \in D$. It follows that for all $(c, d) \in C \times D$ and $(a, b) \in A \times B$, we have

$$\begin{aligned} (a,b) \cdot (c,d) \cdot (a,b)^{-1} &= (a,b) \cdot (c,d) \cdot (a^{-1}, b^{-1}) \\ &= (aca^{-1}, bdb^{-1}) \in C \times D. \end{aligned}$$

Hence, $(C \times D) \triangleleft (A \times B)$.

b) Consider the surjective group homomorphism

$$\begin{aligned} \phi :& A \times B \to (A / C) \times (B / D), \\ & (a,b) \mapsto (aC, bD). \end{aligned}$$

The kernel of ϕ is $C \times D$, and the result follows from the 1st isomorphism theorem.

2. For $n \geq 3$, show that A_n contains a subgroup isomorphic to S_{n-2}.

Proof.

Fix $\alpha \in S_n$, which is the transposition $(n-1, n)$. Consider the map

$$\phi : S_{n-2} \to A_n,$$

where

$$\phi(\sigma) = \begin{cases} \sigma & \text{if } \sigma \text{ is even,} \\ \sigma\alpha & \text{if } \sigma \text{ is odd.} \end{cases}$$

The map ϕ is a group homomorphism, and $\ker(\phi) = \{e\}$, so $\operatorname{im}(\phi)$ is isomorphic to S_{n-2}.