I. True/False

1. The binary operation on a group is completely determined by the identity element and inverse operation.  \textbf{False}

2. If \( \phi : G \to G' \) is a group homomorphism and \( H \) is a subgroup of \( G \), then \( \phi(H) \) is a subgroup of \( G' \).  \textbf{True}

3. An inner automorphism of an abelian group must be the identity map.  \textbf{True}

4. If \( H \) and \( K \) are subgroups of \( G \) such that \( H \cap K = \{e\} \) and \( G = HK \), then \( H \) is a normal subgroup of \( G \).  \textbf{False}
   
   \text{Counter example: } G = S_3, H = \langle (23) \rangle \text{ and } K = \langle (123) \rangle.

5. Among all abelian groups of order \( p^n \) for some prime number \( p \), the one with the minimum number of subgroups, is the cyclic one.  \textbf{True}

II. Short Answers

1. In each of the following cases, indicate whether the given subgroup \( H \) of \( S_4 \) is normal. If \( H \) is normal, state what the quotient group \( S_4/H \) is.
   
   a) \( H = \{e\} \)
   
   \{e\} is normal. \( S_4 / \{e\} \cong S_4 \).

   b) \( H = \langle (12) \rangle \)
   
   \langle (12) \rangle \text{ is not normal.}

   c) \( H = S_3 \)
   
   \( S_3 \) is not normal.

   d) \( H = A_4 \)
   
   \( A_4 \) is normal. \( S_4 / A_4 \cong \mathbb{Z}/2\mathbb{Z} \).

   e) \( H = \langle (1234), (14)(23) \rangle \)
   
   \( H \) is not normal.

2. In each of the following cases, indicate the order of the element \( g \) in the given group \( G \).
   
   a) \( g = m, \quad G = \mathbb{Z}/n\mathbb{Z} \).
   
   \( m \) \text{ has order } \frac{n}{\gcd(m,n)}.

   b) \( g = (52743), \quad G = S_8 \).
   
   \text{The order of } (52743) \text{ is } 5.

   c) \( g = 5 + \langle 4 \rangle, \quad G = (\mathbb{Z}/12\mathbb{Z}) / \langle 4 \rangle. \)
   
   \text{The order of } 5 + \langle 4 \rangle \text{ is } 4.

   d) \( g = (2,1) + \langle (4,4) \rangle, \quad G = (\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}) / \langle (4,4) \rangle. \)
   
   \text{The order of } (2,1) + \langle (4,4) \rangle \text{ is } 4.

   e) \( g = (12345)(5678)(89), \quad G = S_9. \)
   
   \text{The order of } (12345)(5678)(89) = (123456789) \text{ is } 9.
III. Proofs

1. Let $C$ be a normal subgroup of $A$, and let $D$ be a normal subgroup of $B$. Prove that
   a) $(C \times D) \trianglelefteq (A \times B)$;
   b) $(A \times B) / (C \times D) \cong (A / C) \times (B / D)$.

   Proof.
   a) Since $C \trianglelefteq A$ and $D \trianglelefteq B$, for all $c \in C$ and $a \in A$, we have $aca^{-1} \in C$; and for all $d \in D$ and $b \in B$, we have $bdb^{-1} \in D$. It follows that for all $(c, d) \in C \times D$ and $(a, b) \in A \times B$, we have
   $$(a, b) \cdot (c, d) \cdot (a, b)^{-1} = (a, b) \cdot (c, d) \cdot (a^{-1}, b^{-1})$$
   $$= (aca^{-1}, bdb^{-1}) \in C \times D.$$ 
   Hence, $(C \times D) \trianglelefteq (A \times B)$.

   b) Consider the surjective group homomorphism
   $$\phi : A \times B \to (A / C) \times (B / D),$$
   $$(a, b) \mapsto (aC, bD).$$
   The kernel of $\phi$ is $C \times D$, and the result follows from the 1st isomorphism theorem.

2. For $n \geq 3$, show that $A_n$ contains a subgroup isomorphic to $S_{n-2}$.

   Proof.
   Fix $\alpha \in S_n$, which is the transposition $(n - 1, n)$. Consider the map
   $$\phi : S_{n-2} \to A_n,$$
   where
   $$\phi(\sigma) = \begin{cases} 
   \sigma & \text{if } \sigma \text{ is even,} \\
   \sigma\alpha & \text{if } \sigma \text{ is odd.}
   \end{cases}$$
   The map $\phi$ is a group homomorphism, and $\ker(\phi) = \{e\}$, so $\im(\phi)$ is isomorphic to $S_{n-2}$.