I. True/False

1. Every finite group admits a composition series. **True**

2. All abelian groups are nilpotent. **False**

For example, $\mathbb{Z}$ is abelian, but not solvable, and therefore not nilpotent.

3. Let X be a G-set, and let $x \in X$. If G is finite, then $|G| = |\mathcal{O}_x| \cdot |G_x|$. **True**

Recall that $\mathcal{O}_x = \{y \in X \mid y = gx \text{ for some } g \in G\}$ and $G_x = \{g \in G \mid gx = x\}$.

4. Every non-trivial subgroup of free group is free. **True**

Theorem 39.10. Note that the proof of this result is not trivial.

5. If G is a free group and $\phi : G \to H$ is a group homomorphism, then the image of ϕ is a free group. **False**

See, for example, Exercises 39.3 and 39.4, which is part of your homework.

II. Short Answers (20 points) No justification needed.

1. Give an example of a solvable group which is not nilpotent.

For example, S_3.

2. Give two examples of non-abelian groups of order 60. Make sure that the two examples are not isomorphic.

For example, D_{30} (or D_{60}) and A_5. Note that A_5 is simple, and D_{30} is not simple, so they are not isomorphic.

3. Give isomorphic refinements to the following two series of groups:

$$\{0\} < 6\mathbb{Z} < 2\mathbb{Z} < \mathbb{Z},$$
$$\{0\} < 35\mathbb{Z} < 5\mathbb{Z} < \mathbb{Z}$$

We have the refinements as follows:

$$\{0\} < 210\mathbb{Z} < 30\mathbb{Z} < 6\mathbb{Z} < 2\mathbb{Z} < \mathbb{Z},$$
$$\{0\} < 210\mathbb{Z} < 70\mathbb{Z} < 35\mathbb{Z} < 5\mathbb{Z} < \mathbb{Z}$$

4. List all groups of order 14 up to isomorphism.

There are two groups of order 14, namely D_7 (or D_{14}) and $\mathbb{Z}/14\mathbb{Z}$.

5. List four non-isomorphic groups of order 8.

There are five groups of order 8, namely $\mathbb{Z}/8\mathbb{Z}$, $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, D_4 (or D_8) and the quaternion group Q_8. You may list any four of these.

6. For each of the following group presentations, determine the given group G.

a) $G = \langle a \mid a^n = 1 \rangle$.

$G \cong \mathbb{Z}/n\mathbb{Z}$.

b) $G = \langle a, b \mid a^n = 1,\ b^m = 1,\ ab = ba \rangle$.

$G \cong (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$.

c) $G = \langle a, b \mid a^n = 1,\ b^2 = 1,\ (ba)^2 = 1\rangle$.
$G \cong D_n$ (or $G \cong D_{2n}$).

d) $G = \langle a_1, a_2, \ldots, a_n \mid a_i^2 = 1,\ a_i a_j = a_j a_i\rangle$.
$G \cong (\mathbb{Z}/2\mathbb{Z})^n$.

e) $G = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_i^2 = 1,\ \sigma_1\sigma_3 = \sigma_3\sigma_1,\ (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_3)^3 = 1\rangle$.
$G \cong S_4$ where $\sigma_1 = (12)$, $\sigma_2 = (23)$ and $\sigma_3 = (34)$.

III. Proofs (20 points) In each of the following problems, give a complete proof.

1. Prove that there are no simple groups of order 144.

Note that $144 = 16 \times 9 = 2^4 \times 3^2$. Let G be a simple group of order 144, and let n_3 be the number of Sylow 3-subgroups of G, i.e. the number of subgroups of order 9. By the Sylow theorems, $n_3 = 1, 4, 16$.

If $n_3 = 4$, then acting by conjugation on these 4 Sylow 3-subgroups, we obtain a group homomorphism from G to S_4. Since 144 does not divides 4!, it follows that this group map has a non-trivial kernel and G is not simple. So $n_3 = 16$.

Now if every pair of Sylow 3-subgroups have trivial intersection, then there are at least $16 \times 8 = 128$ elements of order 3 or 9 in G. This leaves only 16 more elements, which means that there are only one Sylow 2-subgroup, necessarily normal, contradiciting the assumption that G is simple.

Thus, there exist two different Sylow 3-subgroups P and Q such that $|P \cap Q| = 3$. Let $N = N_G(P \cap Q)$ be the normalizer of $P \cap Q$, which is the largest subgroup of G that contain $P \cap Q$ as a normal subgroup. If $N = G$, then $P \cap Q$ is normal in G and G is not simple. Since N contain both P and Q, $|N| > 9$ and is divisible by 9, so $|N| = 18, 36, 72$. By the Sylow theorems, groups of order 18 contain only one group of order 9, contradicting the assumption that $P \neq Q$. On the other hand, if $|N|$ is either 36 and 72, then $[G : N]$ is either 2 or 4, and the action of G on the cosets of N yields group homomorphisms from G to S_2 or S_4. These maps have non-trivial kernels, contradicting the assumption that G is simple.

2. Recall that for a group G, the commutator subgroup $G^{(1)}$ is generated by the collection of commutators $\{ghg^{-1}h^{-1} \mid g, h \in G\}$.

Show that if N is normal subgroup of G, then $N^{(1)}$ is also a normal subgroup of G.

We need to show that for every $g \in G$ and every $x \in N^{(1)}$, we have $gxg^{-1} \in N^{(1)}$. Now x is a product of commutators of N, so by inserting gg^{-1} between each product of commutators, it suffices to show that for every commutator $n_1 n_2 n_1^{-1} n_2^{-1}$ where $n_1, n_2 \in N$, we have $g n_1 n_2 n_1^{-1} n_2^{-1} g^{-1}$ is again a commutator.

Indeed, since N is normal in G, it follows that $g n_1 g^{-1} = n_3$ and $g n_1 g^{-1} = n_4$ for some $n_3, n_4 \in N$. Thus,

$$\begin{aligned} g n_1 n_2 n_1^{-1} n_2^{-1} g^{-1} &= \left(g n_1 g^{-1}\right)\left(g n_2 g^{-1}\right)\left(g n_1^{-1} g^{-1}\right)\left(g n_2^{-1} g^{-1}\right) \\ &= n_3 n_4 n_3^{-1} n_4^{-1}, \end{aligned} \tag{1}$$

as required.