Name:

1. Let \( f : M \to M \) be a diffeomorphism. For vector fields \( X \) and \( Y \) on \( M \), show that
\[
f_*([X,Y]) = [f_*(X),f_*(Y)]
\]

2. Consider two vector fields \( X = \frac{\partial}{\partial y} \) and \( Y = y \frac{\partial}{\partial x} - \frac{\partial}{\partial z} \) on \( \mathbb{R}^3 \), where we use the coordinates \((x,y,z)\). Is it possible to find a 2-dimensional submanifold of \( \mathbb{R}^3 \) with the property that both \( X \) and \( Y \) are tangent to it at all its points? Justify your answer.

3. Let \( V \) be a finite dimensional vector space. Then \( TV = V \times V \).

a) The trivial map \( E : x \mapsto (x,x) \) defines a section of the tangent bundle, i.e. a vector field. Compute the time-\( t \) flow of this vector field and determine whether it is complete or not.

b) If \( A : V \to V \) is a linear map, then the map \( A : x \mapsto (x, Ax) \) defines a vector field on \( V \). Compute its flow and determine if it is complete.

c) If \( A \) and \( B \) are two linear maps, compute the Lie derivative of the vector fields determined by \( A \) and \( B \), i.e. compute their bracket. Verify that if the vector fields commute, then the flows commute.

4. a) Let \( M \) be a smooth manifold, and let \( \iota : L \hookrightarrow M \) be an embedding. Show that \( TL \) is a subbundle of \( TM|_L := \iota^*TM \).

For your convenience, here is the definition of subbundle.

**Definition 1.** Given a vector bundle \( \pi_E : E \to M \), a subbundle of \( E \) is a vector bundle \( \pi_F : F \to M \) such that \( F \) is a embedded submanifold of \( E \), \( \pi_F \) is the restriction of \( \pi_E \) to \( F \), and for each \( p \in M \),
\[
F_p = F \cap E_p
\]
is a linear subspace of \( E_p \).

b) Now use the vector space quotient to define the notion of vector bundle quotient, and hence the notion of normal bundle.

c) Give an example of a closed embedded submanifold of a compact manifold such that the normal bundle is trivial. Prove your claim.

d) Give an example of a closed embedded submanifold of a compact manifold such that the normal bundle is not trivial.

**Bonus.** Prove the claim in d).

5. Let \( M \) be a compact smooth manifold, and let \( E \to M \) be a vector bundle of rank \( k \). Prove that \( E \) admits a section \( s \) with the following property:

a. if \( k > \dim M \), then \( s \) is nowhere vanishing;

b. if \( k \leq M \), then the set of points where \( s \) vanishes is a compact codimension \( k \) submanifold of \( M \).

Hint: Use transversality.

Remark: In particular, \( M \) admits a vector field that vanishes at finitely many points.