ANSWERS TO QUIZ 6

Show your work not just your final answer

(1) Let $M_2$ be the vector space of $2 \times 2$ matrices with the inner product defined as usual, which I recall. For $A, B \in M_2$, 
\[ \langle A, B \rangle = \text{tr}(A^T B) \] 
where tr stands for the trace. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $W$ be the subspace spanned by $I, A$. Calculate the orthogonal projection of $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to $W$.

Solution. There are two ways to do this. One is to find an orthogonal basis for $W$ using Gram-Schmidt process and then appealing to the formula for projection. (The formula does not apply without changing into an orthogonal basis, a mistake many of you have made in homework.) The other method is to note that $\text{proj}_W E = aI + bA$ for some scalars $a, b$ and $E - aI - bA$ is orthogonal to both $I$ and $A$ to solve for $a, b$. We will follow the latter here.

So, we calculate
\[ \langle I, E - aI - bA \rangle = \text{tr}(E - aI - bA) = 2 - 2a - 2b = 0. \]
\[ \langle A, E - aI - bA \rangle = \text{tr}(A^T E - aA^T - bA^T A) = 2 - 2a - 4b = 0. \]
This gives $a = 1, b = 0$ and so $\text{proj}_W(E) = I$. □

(2) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree at most 2. Define an inner product by, $\langle p(t), q(t) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$. Starting with the standard basis $1, t, t^2$, use Gram-Schmidt process to find an orthogonal basis.

Solution. Gram-Schmidt says, keep 1 as it is and change first $t$ to $t - \frac{\langle 1, t \rangle}{\langle 1, 1 \rangle} 1$. $\langle 1, t \rangle = 3$ and $\langle 1, 1 \rangle = 3$, thus our second basis is $u = t - 1$. Next, we change $t^2$ to, $t^2 - \frac{\langle 1, t^2 \rangle}{\langle 1, 1 \rangle} 1 = \frac{\langle 1, t^2 \rangle}{\langle u, u \rangle} (t - 1)$. One easily calculates, $\langle 1, t^2 \rangle = 5, \langle u, t^2 \rangle = 4, \langle u, u \rangle = 2$. Thus, we may take the third to be $v = t^2 - \frac{5}{3} - 2(t - 1) = t^2 - 2t + \frac{1}{3}$.
Thus an orthogonal basis is,

\[ 1, t - 1, t^2 - 2t - \frac{1}{3} \]

\[ \Box \]

(3) Show that for a positive integer \( n \),

\[ \int_0^1 x^n e^x dx \leq \sqrt{\frac{e^2 - 1}{4n + 2}}. \]

(Hint: Use Cauchy-Schwartz inequality)

**Solution.** Use Cauchy-Schwarz inequality. We consider the vector space \( C^0[0, 1] \), set of continuous function on the unit closed interval with the inner product \( \langle f, g \rangle = \int_0^1 fg dx \). Then, by CS inequality,

\[
\int_0^1 x^n e^x dx = \langle x^n, e^x \rangle \\
\leq \sqrt{\langle x^n, x^n \rangle \langle e^x, e^x \rangle} \\
= \sqrt{\int_0^1 x^{2n} dx \int_0^1 e^{2x} dx} \\
= \sqrt{\frac{1}{2n+1} \frac{e^2 - 1}{2}} \\
= \sqrt{\frac{e^2 - 1}{4n + 2}}
\]

\[ \Box \]

(4) Find an orthogonal matrix \( P \) and a diagonal matrix \( D \) such that \( PDP^T = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} \).

**Solution.** We first calculate the eigenvalues. The characteristic polynomial is \( \lambda^2 - 2\lambda - 24 = (\lambda - 6)(\lambda + 4) \). So, the eigenvalues are 6, -4. Easy to check that unit eigenvector corresponding to 6 is \( \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \) and the one corresponding to -4
is, \[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\end{bmatrix}
\]. Thus,

\[
P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}
\]

\[\Box\]

(5) Let \(B\) be an \(n \times n\) symmetric matrix with \(B^2 = B\). For any vector \(x \in \mathbb{R}^n\), show that \(((x - Bx) \cdot Bx) = 0\).

\textit{Solution.} We calculate:

\[
((x - Bx) \cdot Bx) = (x \cdot Bx) - (Bx \cdot Bx)
\]

\[
= x^T Bx - (Bx)^T Bx
\]

\[
= x^T Bx - x^T B^T Bx
\]

\[
= x^T Bx - x^T Bx = 0
\]

The last step, since \(B^T B = BB = B^2 = B\). \[\Box\]

(6) Find a \(3 \times 3\) symmetric matrix \(A\) so that for any vector \(x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\), \(x^T Ax = 4x_2^2 - 2x_1x_2 + 4x_2x_3\).

\textit{Solution.} See example 2 on page 403.

\[
A = \begin{bmatrix}
0 & -1 & 0 \\
-1 & 0 & 2 \\
0 & 2 & 4
\end{bmatrix}
\]

\[\Box\]

(7) Find a change of coordinates \(x = Py\) so that the quadratic form \(2x_1^2 + 6x_1x_2 - 6x_2^2\) has no cross product terms.

\textit{Solution.} As before, the associated matrix is \(A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\).

Its characteristic polynomial is \(\lambda^2 + 4\lambda - 21\) and so the eigenvalues are 3, -7. Then, one can calculate corresponding unit
eigenvectors as
\[ u = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}, \quad v = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}. \]

Taking \( P = [u \ v], \) and \( D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}, \) we have \( P^TAP = D. \)
So, if \( x = Py, \) we get,
\[ y^TDy = y^TP^TAPy = x^TAx. \]
The last term corresponds to our quadratic form and the first term corresponds to the quadratic form \( 3y_1^2 - 7y_2^2. \)

(8) Let \( A \) be an \( n \times n \) invertible matrix. Show that \( A^TA \) is positive definite.

Solution. See Theorem 5 on page 407. Also see page 418.
According to the above theorem, we only need to show that the eigenvalues of \( B = A^TA \) are all positive. So, let \( \lambda \) be an eigenvalue of \( B \) and \( u \) be a corresponding eigenvector. Then we have \( Bu = \lambda u. \) This gives, writing \( v = Au \) and noting \( v \neq 0 \) since \( A \) is invertible,
\[ 0 < v^Tv = u^T A^TAu = \lambda u^Tu \]
This says \( \lambda \) multiplied by a positive number is positive and thus \( \lambda \) must be positive.

(9) Find the singular values of the matrix \( A = \begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix}. \)

Solution. \( A^TA = \begin{bmatrix} 73 & 24 \\ 24 & 9 \end{bmatrix} \) and its characteristic polynomial is \( (\lambda - 1)(\lambda - 81). \) So, the singular values of \( A \) are 9, 1.

(10) Find a singular value decomposition for the matrix,
\[ A = \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} \]
Solution. We first calculate
\[
A^TA = \begin{bmatrix} 7 & 5 & 0 \\ 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}
\]

One next calculates its characteristic polynomial to be \(\lambda^2 - 100\lambda + 900\) and thus its eigenvalues are \(90 \geq 10\). So, the singular values are \(\sigma_1 = 3\sqrt{10} \geq \sigma_2 = \sqrt{10}\). Thus the rank of \(A = 2\).

Next we calculate the corresponding unit eigenvectors to be \(v_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, v_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}\). Thus, we have,
\[
V = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \end{bmatrix}
\]

Finally, we take \(u_1 = \frac{Av_1}{3\sqrt{10}}, u_2 = \frac{Av_1}{\sqrt{10}}\). So,
\[
u_1 = \begin{bmatrix} 15/\sqrt{5} \\ 15/\sqrt{5} \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 5/\sqrt{5} \\ -5\sqrt{5} \\ 0 \end{bmatrix}.
\]

We extend these to an orthonormal basis for \(\mathbb{R}^3\) and one can see that \(e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\) is one such. Thus, we may take,
\[
U = \begin{bmatrix} 15/\sqrt{5} & 5/\sqrt{5} & 0 \\ 15/\sqrt{5} & -5\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Then we have \(A = U\Sigma V^T\). \(\square\)