(1) Let \( f, g, h \) be functions from \( \mathbb{R} \) to itself which are linearly independent in the vector space of all such functions from reals to itself. Define elements of the vector space of signals by \( F(k) = f(k), G(k) = g(k), H(k) = h(k), k \in \mathbb{Z} \). Are \( F, G, H \) linearly independent?

(2) Show that if \( P \) is a stochastic matrix, then so is \( P^2 \).

(3) Decide which of the following are true (with justifications always).
   (a) If \( A \) is a square matrix and \( \lambda, \mu \) are two eigenvalues of \( A \) then \( \lambda \mu \) is an eigenvalue of \( A^2 \).
   (b) An eigenvalue of a matrix \( A \) is also an eigenvalue of \( A^T \).
   (c) An \( n \times n \) matrix with \( n \) linearly independent eigenvectors is invertible.

(4) Let \( p(t) = c_0 + c_1 t + \cdots + c_n t^n \) be a polynomial and \( A \) a square matrix. Define \( p(A) \) to be \( c_0 I + c_1 A + \cdots + c_n A^n \). Show that if \( \lambda \) is an eigenvalue of \( A \), then \( p(\lambda) \) is an eigenvalue of \( p(A) \).

(5) Let

\[
A = \begin{bmatrix}
0.4 & -0.3 \\
0.4 & 1.2
\end{bmatrix}
\]

Calculate \( \lim_{k \to \infty} A^k \).

(6) If \( A \) is an \( n \times n \) real matrix such that \( A = A^T \) (such matrices are called symmetric), show that its eigenvalues are real.

(7) Find an invertible matrix \( P \) with main diagonal elements 2, 1, and a matrix \( C \) of the form

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\]

such that

\[
PCP^{-1} = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}
\]
(8) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given as $T(x) = Ax$, where

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

Find an ordered basis $B$ of $\mathbb{R}^3$, with all coordinates being integers, so that the matrix of $T$ relative to $B$ (for both $\mathbb{R}^3$) is diagonal.

(9) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that the characteristic polynomial is $p(t) = \lambda^2 - (a + d)\lambda + \det A$. Then show that $p(A) = A^2 - (a + b)A + \det AI = 0$.

(10) Find all real eigenvalues and the corresponding eigenvectors for

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

(11) Find the steady-state vector for

$$\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$$

(12) Decide which of the following are true (with justifications)

(a) If $A, B$ are two $n \times n$ matrices and $\lambda$ is an eigenvalue for both, then it is an eigenvalue of $A + B$.

(b) If $A, B$ are two $n \times n$ matrices and $x$ is an eigenvector for both, then $x$ is also an eigenvector for $A + B$.

(c) If $A$ is an $n \times n$ matrix, then $A - I_n$ is invertible if 1 is not an eigenvalue of $A$. 