Mathematics 309: Matrix Algebra
Model Solutions to the Midterm Examination

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No collaboration, calculators, notes, texts, problem sets, or worked solutions are permitted. Please write your complete answers in the blue notebooks. You have until the end of class to answer all of the following problems.

Problem 1: For the following matrices A and vectors $\mathbf{b}$, determine whether the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. If it is, determine all solutions $\mathbf{x}$.

$$(\mathbf{i})\quad A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix};\quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix};\qquad (\mathbf{ii})\quad A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix};\quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Solution 1: (**i**) Inconsistent: third equation is $0x_1 + 0x_2 = 1$, which has no solution $\mathbf{x} = (x_1, x_2)$.

(**ii**) Consistent, since the matrix A is in row echelon form and each zero row has a corresponding zero in the right-hand side vector $\mathbf{b}$. Each column contains a leading nonzero (in fact, a leading one), so the solution is unique and may be found by back substitution: $1x_2 = 1 \implies x_2 = 1$; $2x_1 = 1 - x_2 = 0 \implies x_1 = 0$; $\mathbf{x} = (x_1, x_2) = (0, 1)$. ■

Problem 2: Prove or find a counterexample to the following statement about $n \times n$ matrices A, B, C: if $AB = AC$ and $A \neq 0$, then $B = C$.

Solution 2: The statement is false. Counterexample: $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $AB = 0 = AC$, but $B \neq C$. ■

Problem 3: Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 7 \\ 0 & 6 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$. Compute the determinants of A, A^2, and A^T.

Solution 3: Exchanging rows 2 and 3 makes A an upper triangular matrix. By theorem 2.1.3 and Summary I on page 110, the determinant of A is $(-1)(1)(6)(5)(10) = -300$. By theorem 2.2.3 of page 112, $\det(A^2) = \det(A)^2 = (-300)^2 = 90,000$. By theorem 2.1.1 of page 105, $\det(A^T) = \det(A) = -300$. ■

Problem 4: Determine the nullspace of the matrix $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{pmatrix}$.

Solution 4: Gauss-Jordan elimination to reduced row echelon form:

$$A \to \begin{pmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, every solution $\mathbf{x} = (x_1, x_2, x_3, x_4)$ to $A\mathbf{x} = 0$ satisfies $x_1 = -x_2 - 5x_4$ and $x_3 = -3x_4$, with two free parameters x_2, x_4. Hence the nullspace of A is $\{(-s-5t, s, -3t, t) : s, t \in \mathbf{R}\} = \text{Span}\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}\}$. ■

Problem 5: Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Suppose that scalars $a, b \in \mathbf{R}$ are given and $\mathbf{x} = a\mathbf{u}_1 + b\mathbf{u}_2$. Find c, d such that $\mathbf{x} = c\mathbf{v}_1 + d\mathbf{v}_2$.

Solution 5: Write $U = (\mathbf{u}_1, \mathbf{u}_2)$ and $V = (\mathbf{v}_1, \mathbf{v}_2)$. Following Example 5 on p.169, the transition matrix from $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{v}_1, \mathbf{v}_2]$ is

$$V^{-1}U = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} -5 & -14 \\ 4 & 11 \end{pmatrix}.$$

Thus $\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -5 & -14 \\ 4 & 11 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -5a - 14b \\ 4a + 11b \end{pmatrix}$, so $c = -5a - 14b$ and $d = 4a + 11b$. ■

Problem 6: Let $\mathbf{u} = e^x$, $\mathbf{v} = xe^x$, and let $E = \text{Span}\{\mathbf{u}, \mathbf{v}\} \subset C[-1, 1]$. Define a linear operator $L : E \to E$ by the formula

$$L(f(x)) = f(x) + f'(x).$$

Find the matrix of L with respect to the ordered basis $[\mathbf{u}, \mathbf{v}]$ of E.

Solution 6: Compute

$$L(\mathbf{u}) = L(e^x) = e^x + (e^x)' = e^x + e^x = 2e^x + 0xe^x = 2\mathbf{u} + 0\mathbf{v};$$
$$L(\mathbf{v}) = L(xe^x) = xe^x + (xe^x)' = xe^x + e^x + xe^x = 1e^x + 2xe^x = 1\mathbf{u} + 2\mathbf{v}.$$

Transposing the array of coefficients that appears at the right gives the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ representing L with respect to the ordered basis $[\mathbf{u}, \mathbf{v}]$ of E. ■