

# Ma 3520: Differential Equations and Dynamical Systems

## Homework Assignment 2

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Read Chapters 3 and 4 of the textbook, “Nonlinear Dynamics and Chaos,” 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

- (Ex.3.1.3, p.88) Suppose that  $x = x(t)$  and  $\dot{x} = r + x - \ln(1 + x)$ .
  - Sketch the qualitatively different phase diagrams as  $r$  is varied,
  - Determine the critical value of  $r$  for which there is a saddle-node bifurcation.
  - Sketch the bifurcation diagram of fixed points  $x^*$  versus  $r$ .
- (Ex.3.2.2, p.89) Suppose that  $x = x(t)$  and  $\dot{x} = rx - \ln(1 + x)$ .
  - Sketch the qualitatively different phase diagrams as  $r$  is varied,
  - Determine the critical value of  $r$  for which there is a transcritical bifurcation.
  - Sketch the bifurcation diagram of fixed points  $x^*$  versus  $r$ .
- (Ex.3.4.3, p.91) Suppose that  $x = x(t)$  and  $\dot{x} = rx - 4x^3$ .
  - Sketch the qualitatively different phase diagrams as  $r$  is varied,
  - Determine the critical value of  $r$  for which there is a pitchfork bifurcation.
  - Sketch the bifurcation diagram of fixed points  $x^*$  versus  $r$ .
- (Ex.3.4.[5,...,10], p.92) For the following dynamical systems, classify the type of bifurcations that arise: saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork.
  - $\dot{x} = r - 3x^2$
  - $\dot{x} = rx - \frac{x}{1+x}$
  - $\dot{x} = 5 - re^{-x^2}$
  - $\dot{x} = rx - \frac{x}{1+x^2}$
  - $\dot{x} = x + \tanh(rx)$
  - $\dot{x} = rx + \frac{x^3}{1+x^2}$
- (Ex.3.6.2, p.96) Consider the system  $\dot{x} = h + rx - x^2$  for constants  $h, r$ . When  $h = 0$  the system undergoes a transcritical bifurcation at  $r = 0$ .
  - Plot the bifurcation diagrams for  $h < 0$ ,  $h = 0$ , and  $h > 0$ .

(b) Sketch the regions in the  $(r, h)$ -plane that correspond to qualitatively different phase diagrams and identify the bifurcations that occur on the boundaries of these regions.

(c) Plot the potentials  $V(x)$  corresponding to the different regions of part (b).

6. (Ex.4.1.[2, . . . , 7], p.127) For each of the following dynamical systems, find and classify all fixed points and sketch the phase portrait on the circle.

(a)  $\dot{\theta} = 1 + 2 \cos \theta$

(b)  $\dot{\theta} = \sin 2\theta$

(c)  $\dot{\theta} = \sin^3 \theta$

(d)  $\dot{\theta} = \sin \theta + \cos \theta$

(e)  $\dot{\theta} = 3 + \cos 2\theta$

(f)  $\dot{\theta} = 3 + \sin k\theta$  where  $k$  is a positive integer.

7. (Ex.4.3.1, p.129) The time to pass through a saddle-node bottleneck is

$$T = \int_{-\infty}^{\infty} \frac{dx}{r + x^2},$$

where  $r > 0$  is small. Evaluate this integral with an appropriate trigonometric substitution.

8. (Ex.4.3.[3, . . . , 8], pp.129–130) For each of the following flows on the circle, draw the phase portraits for qualitatively different values of the control parameter  $\mu$ .

(a)  $\dot{\theta} = \mu \sin \theta - \sin 2\theta$

(b)  $\dot{\theta} = \frac{\sin \theta}{\mu + \cos \theta}$

(c)  $\dot{\theta} = \mu + \cos \theta + \cos 2\theta$

(d)  $\dot{\theta} = \mu + \sin \theta + \cos 2\theta$

(e)  $\dot{\theta} = \frac{\sin \theta}{\mu + \sin \theta}$

(f)  $\dot{\theta} = \frac{\sin 2\theta}{1 + \mu \sin \theta}$

9. (Ex.4.4.1, p.130) Find the conditions on the parameters  $b, m, g, L$  for which it is valid to approximate

$$mL^2\ddot{\theta} + bL^2\dot{\theta} + mgL \sin \theta = \Gamma$$

by its overdamped limit  $bL^2\dot{\theta} + mgL \sin \theta = \Gamma$ . Justify your answer.