Ma 3520: Differential Equations and Dynamical Systems
Homework Assignment 4

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Read Chapter 7 and 8 of the textbook, "Nonlinear Dynamics and Chaos," 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

1. (Ex.7.1.8abc, p.251) Consider the nonlinear oscillator whose governing equation is

$$\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0, \qquad a > 0.$$

   (a) Find and classify all the fixed points.

   (b) Show that the system has a circular limit cycle, and find its amplitude.

   (c) Determine the stability of the limit cycle.

2. (Ex.7.2.7, p.253) Consider the system $\dot{x} = y + 2xy \overset{\text{def}}{=} f(x, y)$, $\dot{y} = x + x^2 - y^2 \overset{\text{def}}{=} g(x, y)$.

   (a) Show that $\partial f/\partial y = \partial g/\partial x$, so that it is a gradient system.

   (b) Find a potential V such that $(\dot{x}, \dot{y}) = -\nabla V(x, y)$. [Hint: use partial integration as in the solution of exact differential equations.]

   (c) Sketch the system's phase portrait, and include the level curves of V.

3. (Ex.7.2.10, p.254) Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

4. (Ex.7.3.3, p.256) Use the Poincaré-Bendixson theorem to show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.

5. (Ex.7.4.1, p.259) Use Liénard's theorem to show that the equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + \tanh x = 0$, with $\mu > 0$ has exactly one periodic solution, and classify its stability.

6. (Ex.8.2.1, p.315) Consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. Find the curves in (μ, a)-space where Hopf bifurcations occur.

7. (Ex.8.2.6, p.316) In the system $\dot{x} = \mu x + y - x^3$, $\dot{y} = \mu y - x - 2y^3$, a Hopf bifurcation occurs at the origin when $\mu = 0$. Plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. [Hint: check your answer using the method of Ex.8.2.12, p.318.]

8. (Ex.8.4.1, p.322) For the system $\dot{r} = r(1 - r^2)$, $\dot{\theta} = \mu - \sin\theta$:

   (a) Let $x = r\cos\theta$ and $y = r\sin\theta$, write the system in terms of x, y.

   (b) For some μ slightly greater that $\mu_c = 1$, sketch the trajectories $x(t), y(t)$ for various initial $x(0), y(0)$. [Hint: use `rk4a()` from the class website.]

   (c) Estimate, by trial and error with different μ, the relationship between $\mu - \mu_c$ and the period of a trajectory near the infinite-period bifurcation at μ_c.

9. (Ex.8.4.2, p.322) What types of global bifurcations occur in the system $\dot{r} = r(\mu - \sin r)$, $\dot{\theta} = 1$ for various values of μ?

10. (Ex.8.4.3, p.322) Sketch the phase portraits of the system $\dot{x} = \mu x + y - x^2$, $\dot{y} = -x + \mu y + 2x^2$, which has a homoclinic bifurcation at $\mu_c \approx 0.066$, for values of μ just above and below μ_c.