Ma 3520: Differential Equations and Dynamical Systems
Homework Assignment 5

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Read Chapters 9 and 10 of the textbook, "Nonlinear Dynamics and Chaos," 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

1. (Ex.9.1.4,p.377) The Maxwell-Bloch equations for a laser are

$$\begin{aligned} \dot{E} &= \kappa(P - E) \\ \dot{P} &= \gamma_1(ED - P) \\ \dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP) \end{aligned}$$

   [NOTE: there is a typo in the textbook at the P equation.]

   (a) Find a fixed point with $E^* = 0$, show that it loses stability above some critical value $\lambda = \lambda_c$, and find λ_c.

   (b) Classify the bifurcation at this λ_c.

   (c) Find a change of variables that maps the system into the Lorenz system.

2. (Ex.9.2.3,p.378) Show that all trajectories of the Lorenz system eventually enter and remain inside a large sphere S of the form

$$V \overset{\text{def}}{=} x^2 + y^2 + (z - r - \sigma)^2 \; = \; C,$$

   for sufficiently large $C > 0$. [HINT: show that V decreases along Lorenz trajectories for all (x, y, z) outside some fixed ellipsoid, then choose C large enough so that S encloses the ellipsoid.]

3. (Ex.9.3.1,p.379) Consider the system $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$ with constants ω_1, ω_2, for $(\theta_1(t), \theta_2(t))$ on the torus $[0, 1) \times [0, 1)$, namely taking values "modulo 1."

   This system is quasiperiodic but not periodic if ω_1/ω_2 is an irrational number. Find the largest Liapunov exponent of this system in that case.

4. (Ex.9.4.1,p.380) Using the codes in `lorenz.txt` on the class website, compute the Lorenz map for $r = 28$, $\sigma = 10$, and $b = 8/3$.

5. (Ex.9.5.[1,2,3],p.381) For $\sigma = 10$, $b = 8/3$, and each of the values of r given below, plot three graphs: $x(t)$, $y(t)$, and $(x(t).z(t))$, for a good range of times $t \geq 0$.

   (a) $r = 166.3$ (intermittent chaos),

   (b) $r = 212$ (noisy chaos),

   (c) selected values $145 < r < 166$ (period doubling).

   [HINT: use the codes in `lorenz.txt` or write your own.]

6. (Ex.10.1.10,p.422) (a) Show that the map $x_{n+1} = 1 + \frac{1}{2}\sin x_n$ has a unique fixed point.

   (b) Determine whether the fixed point is stable.

7. (Ex.10.1.11,p.422) Consider the map $x_{n+1} = 3x_n - x_n^3$.

   (a) Find all fixed points and classify them by stability.

   (b) Draw a cobweb starting at $x_0 = 1.9$. Then prove that x_n remains bounded for all $n = 1, 2, \dots$.

   (c) Draw a cobweb starting at $x_0 = 2.1$. Then prove that $|x_n| \to \infty$ as $n \to \infty$.

8. (Ex.10.1.12,p.423) Newton's method finds the root of an equation $g(x) = 0$ for a differentable function g by iteration of the *Newton map*

$$f(x) = x - \frac{g(x)}{g'(x)}.$$

   (a) Apply the method to the function $g(x) = x^2 - 4$.

   (b) Show that the Newton map in part (a) has stable fixed points $x^* = \pm 2$.

9. (Ex.10.2.2,p.423) Use a cobweb to show that $x^* = 0$ is a globally stable fixed point for the logistic map with $0 \leq r \leq 1$.

10. (Ex.10.2.3,p.423) Compute an orbit diagram for the logisitic map over the range $3.4 \leq r \leq 3.7$ using increments in r chosen to make it look nice.