Ma 3520: Differential Equations and Dynamical Systems
Solutions to Homework Assignment 1

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Read Chapters 1 and 2 of the textbook, "Nonlinear Dynamics and Chaos," third edition, by Steven H.Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

1. (Ex.2.1.2, p.38) At what points x does the flow $\dot{x} = \sin x$ have the greatest velocity to the right?

   Solution: Maximum velocity to the right occurse at those x for which $\dot{x} = \sin x = +1$, namely at $x = 2k\pi + \frac{\pi}{2}$ for integer k. □

2. (Ex.2.2.3, p.39) For the equation $\dot{x} = x - x^3$, sketch the phase portrait, the vector field on the line, the fixed points classified by stability, and trajectories $x(t)$ for various initial values $x(0)$. Then solve the equation analytically.

   Solution: Phase portrait: this is depicted in Figure 1 with the fixed points and flow directions marked.

   Fixed points:
$$0 = \dot{x} = f(x) = x - x^3 = x(1-x)(1+x) \Rightarrow x^* \in \{0, 1, -1\}$$

   Determine stability from the derivative $f'(x) = 1 - 3x^2$. At the fixed points it satisfies:

   - $f'(0) = 1 > 0$ (unstable),
   - $f'(1) = -2 < 0$ (stable), and
   - $f'(-1) = -2 < 0$ (stable).

   Analytic solution: this differential equation is solvable by separation of variables and partial fraction expansion:
$$\frac{dx}{dt} = x(1-x)(1+x) \iff dt = \frac{dx}{x(1-x)(1+x)} = \left[\frac{1}{x} + \frac{1/2}{1-x} - \frac{1/2}{1+x}\right] dx$$

   so
$$t + C = \log|x| - \frac{1}{2}\log|1-x| - \frac{1}{2}\log|1+x|, \quad \Rightarrow Ce^t = \frac{x}{\sqrt{|1-x^2|}}.$$

   For $|x(0)| < 1$, the solution satisfies $|x(t)| < 1$ for all $t > 0$ since $x = \pm 1$ are fixed points. Then the substitution $x \leftarrow \sin y$ yields $x/\sqrt{1-x^2} = \sin y / \cos y = \tan y$, so $Ce^t = \tan y$, so
$$x(t) = \sin\left(\tan^{-1}(Ce^t)\right); \qquad C = \tan(\sin^{-1} x(0)).$$

For $|x(0)| > 1$, the solution satisfies $|x(t)| > 1$ for all $t > 0$ since $x = \pm 1$ are fixed points. Then the substitution $x \leftarrow \pm \sec y$ (depending on the sign of x) yields $x/\sqrt{x^2-1} = \sec y / \tan y = \csc y$, so $Ce^t = \csc y$, so

$$x(t) = \pm \sec\left(\csc^{-1}(Ce^t)\right); \qquad C = \csc(\sec^{-1} x(0)).$$

Trajectories for various initial $x(0) \in [-2, 2]$ are shown in Figure 1. □

3. (Ex.2.2.9, p.39) Find an equation $\dot{x} = f(x)$ whose trajectories resemble those in Fig.2 on p.40 of our textbook.

Solution: One example is the negative logistic equation $\dot{x} = x(x-1)$ with stable fixed point $x^* = 0$ and unstable fixed point $x^* = 1$. □

4. (Ex.2.3.1, p.42) Find the analytic solution to the logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right); \qquad N(0) = N_0$$

for arbitrary initial values N_0 using these two methods:

(a) Use separation of variables.

(b) Substitute $N = 1/x$ and solve the resulting initial value problem for x.

Solution: (a) By separation of variables and expansion into partial fractions, get

$$\left(\frac{1}{N} + \frac{1}{K-N}\right) dN = r\,dt, \quad \Rightarrow N(t) = \frac{K}{1 + Ce^{-rt}}$$

for some constant C. Solving for $C = \frac{K}{N_0} - 1$ using $N(0) = N_0$ gives

$$N(t) = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}}$$

(b) Put $N = 1/x$ to get the differential equation $\dot{x} = -r(x - 1/K)$, which may be solved by separation of variables:

$$\frac{dx}{x - 1/K} = -r\,dt, \quad \Rightarrow x = Ce^{-rt} + \frac{1}{K},$$

for some constant C. Solving for $C = \frac{1}{N_0} - \frac{1}{K}$ from $x(0) = 1/N_0$ gives

$$x(t) = \left(\frac{1}{N_0} - \frac{1}{K}\right)e^{-rt} + \frac{1}{K},$$

which in the original coordinates is

$$N(t) = \frac{1}{\left(\frac{1}{N_0} - \frac{1}{K}\right)e^{-rt} + \frac{1}{K}} = \frac{K}{\left(\frac{K}{N_0} - 1\right)e^{-rt} + 1}$$

just as by the method in part a.

□

5. (Ex.2.4.2, p.45) Use linear stability analysis to classify the fixed points of the system $\dot{x} = x(1-x)(2-x)$.

   Solution: Fixed points are roots of $f(x) = x(1-x)(2-x)$, namely $x^* = 0, 1, 2$. Classify them by determining the sign of $f'(x^*)$, where

   $$f(x) = x^3 - 3x^2 + 2x \qquad \Rightarrow f'(x) = 3x^2 - 6x + 2.$$

   $x^* = 0$ is **unstable** since $f'(0) = 2 > 0$.

   $x^* = 1$ is **stable** since $f'(1) = -1 < 0$.

   $x^* = 2$ is **unstable** since $f'(2) = 2 > 0$. □

6. (Ex.2.4.7, p.45) Use linear stability analysis to classify the fixed points of the system $\dot{x} = ax - x^3$ for fixed $a \in \mathbf{R}$.

   Solution: Put $f(x) = ax - x^3$ and compute $f'(x) = a - 3x^2$.

   If $a = 0$, the only fixed point is $x^* = 0$. Since $f'(0) = 0$, linear stability analysis is inconclusive, but graphical methods indicate that it is **stable**.

   If $a > 0$, write $f(x) = ax - x^3 = x(a - x^2)$ to see that $x^* = 0, \pm\sqrt{a}$. Check $f'(0) = a > 0$ so 0 is **unstable**, but $f'(\pm\sqrt{a}) = -2a < 0$ so $\pm\sqrt{a}$ are both **stable**.

   If $a < 0$, write $f(x) = ax - x^3 = x(a - x^2)$ to see that $x^* = 0$ is the only fixed point, and $f'(0) = a < 0$ so 0 is **stable**. □

7. (Ex.2.5.2, p.46) Show that solutions to $\dot{x} = 1 + x^{10}$ blow up in finite time.

   Solution: Since $\dot{x} = 1 + x^{10} > 1$, the solution from any initial condition will satisfy $x(t_0) > 1$ at some finite $t_0 > 0$ and at all times thereafter. We may therefore assume WOLOG that $x(0) \geq 1$.

   Following the hint, suppose that $\dot{y} = 1 + y^2$ has the same initial condition $y(0) = x(0) \geq 1$.

   Since $\dot{x}(t) > 0$ and $\dot{y}(t) > 0$, both x and y are increasing functions of t with x increasing faster, so $x(t) \geq y(t)$ for all $t \geq 0$.

   But $y(t) = \tan(t + c)$ for some constant c (solved by separation of variables), so $y(t) \to \infty$ as $t \to (2k + \frac{1}{2})\pi - c$ (here k is the smallest integer such that $(2k + \frac{1}{2})\pi > c$), so $x(t) \to \infty$ in finite time as well. □

8. (Ex.2.5.4, p.46) Show that $\dot{x} = x^{1/3}$ has infinitely many solutions $x(t), t \geq 0$ satisfying $x(0) = 0$.

   Solution: Evidently $x_1(t) = 0$ for all $t \geq 0$ is a solution, as is

   $$x_2(t) = \left(\frac{2}{3}t\right)^{3/2}.$$

   Note that for any constant $a > 0$,

   $$\dot{x}_2(t-a) = \frac{d}{dt}\left(\frac{2}{3}[t-a]\right)^{3/2} = \left(\frac{2}{3}[t-a]\right)^{1/2} = x_2(t-a)^{1/3},$$

   and also $x_2(t-a) = 0 = x_1(a)$ at $t = a$. Thus, the function defined by

   $$x(t) = \begin{cases} x_1(t), & 0 \leq t \leq a, \\ x_2(t-a), & t > a \end{cases}$$

   solves $\dot{x} = x^{1/3}$, and there are infinitely many such solutions, each with initial value $x(0) = 0$. □

9. (Ex.2.7.3, p.48) Plot the potential function and classify the equilibrium points of $\dot{x} = \sin x$.

   Solution: Potential function $V(x)$ satisfies $-V'(x) = \sin x$, so $V(x) = \cos x$.

   Equilibrium points are $\{x^* = k\pi : k \in \mathbf{Z}\}$, the crests and troughs of the cosine function. The troughs $(2k+1)\pi$ (odd multiples of π) are **stable**, while the crests $2k\pi$ (even multiples) are **unstable**. □

10. (Ex. 2.8.2, part(d), p.48) Sketch the slope field for $\dot{x} = \sin x$ and draw a few trajectories.

    Solution: Use the following Octave/MATLAB commands:

```
ts=0:1:10; xs=-7:1:7; [t,x]=meshgrid(ts,xs);
h=0.2; dt=h*ones(size(t)); dx=h*sin(x); quiver(t,x,dt,dx);
```

    This produces the slope field in Figure 2 below.

    Trajectories may be plotted by hand. Those starting at $x(0) \in (0, 2\pi)$ will converge to the stable fixed point $x^* = \pi$, while $x(0) \in (-2\pi, 0)$ gives trajectories converging to $x^* = -\pi$, as $t \to \infty$. There are also unstable fixed points $x^* = 0, \pm 2\pi$. □

11. (Ex. 2.8.8, p.49) Use Taylor series to show that the improved Euler method (Heun's method) has local error $O(\Delta t^3)$.

    Solution: Compute the local one step error from exact using Taylor's theorem with three terms:

$$
\begin{aligned}
x_{n+1} - x(t_n + \Delta t) \quad &= \quad x_n + \frac{\Delta t}{2}\left[f(x_n) + f(x_n + \Delta t f(x_n))\right] \\
&\quad - \left[x(t_n) + \Delta t x'(t_n) + \frac{\Delta t^2}{2} x''(t_n) + O(\Delta t^3)\right]
\end{aligned}
$$

    Now $x_n = x(t_n)$ and $x'(t_n) = f(x(t_n)) = f(x_n)$, since we start from an exact solution value at t_n. Also,

$$f(x_n + \Delta t f(x_n)) = f(x_n) + \Delta t f(x_n) f'(x_n) + O(\Delta t^2)$$

    and

$$x''(t_n) = \frac{d}{dt} f(x(t_n)) = f'(x(t_n)) x'(t_n) = f'(x(t_n)) f(x(t_n)) = f'(x_n) f(x_n).$$

    After substitution and cancellation we are left with

$$x_{n+1} - x(t_n + \Delta t) = O(\Delta t^3)$$

    as claimed. □

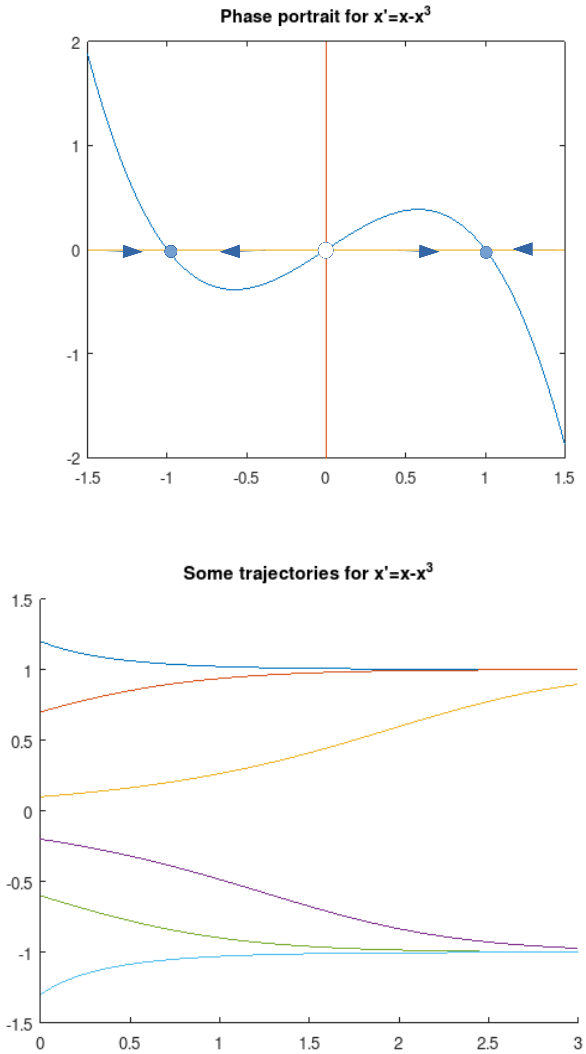

Figure 1: Graphs for Exercise 2.

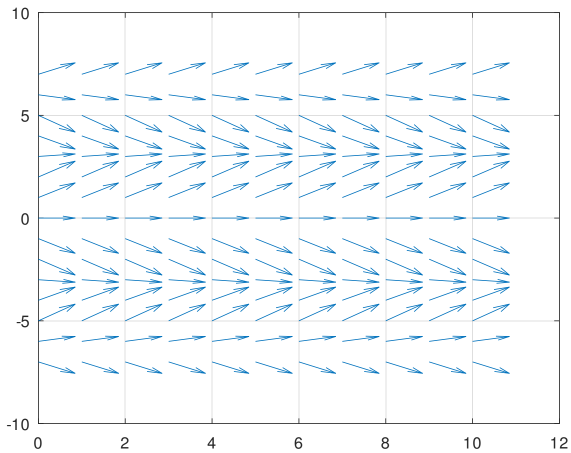

Figure 2: Slope field for $\dot{x} = \sin x$.