Ma 3520: Differential Equations and Dynamical Systems
Solutions to Homework Assignment 2

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Read Chapters 3 and 4 of the textbook, "Nonlinear Dynamics and Chaos," 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

1. (Ex.3.1.3, p.88) Suppose that $x = x(t)$ and $\dot{x} = r + x - \ln(1 + x)$.

   (a) Sketch the qualitatively different phase diagrams as r is varied,

   (b) Determine the critical value of r for which there is a saddle-node bifurcation.

   (c) Sketch the bifurcation diagram of fixed points x^* versus r.

   Solution: (a) See the graphs in Fig.1 below of the two parts $x+r$ and $\ln(1+x)$ of the phase portrait. Their difference is the function $f(x)$ in $\dot{x} = f(x)$. Three qualitatively difference diagrams are:

   - $r > 0$: No fixed points x^* since $f(x) > 0$ for all x.
   - $r = 0$: Semistable fixed point $x^* = 0$.
   - $r < 0$: Two fixed points $x_1^* < 0$ (unstable) and $x_2^* > 0$ (stable).

   (b) At $r = 0$ there is a saddle node bifurcation.

   (c) Use the Taylor approximation $\ln(1 + x) \approx x - x^2/2$ to get

   $$r + x^* = \ln(1 + x^*) \approx x^* - \frac{(x^*)^2}{2} \Rightarrow x^* \approx \pm\sqrt{-2r}.$$

   See the plot in Fig.1 below for the bifurcation diagram depicting these fixed points.

   NOTE: graphs were produced by wxMaxima, arranged using LibreOffice Draw, and exported to PDF for inclusion in this LaTeX file. □

2. (Ex.3.2.2, p.89) Suppose that $x = x(t)$ and $\dot{x} = rx - \ln(1 + x)$.

   (a) Sketch the qualitatively different phase diagrams as r is varied,

   (b) Determine the critical value of r for which there is a transcritical bifurcation.

   (c) Sketch the bifurcation diagram of fixed points x^* versus r.

   Solution: (a) See the graphs in Fig.2 below of the phase portrait $\dot{x} = f(x) = rx - \ln(1 + x)$. Four qualitatively different diagrams are:

   - $r \leq 0$: One stable fixed point $x^* = 0$.
   - $0 < r < 1$: Two fixed points $x_1^* = 0$ (stable) and $x_2^* > 0$ (unstable).

- $r = 1$: Semistable fixed point $x^* = 0$.
- $r > 1$: Two fixed points $x_1^* < 0$ (stable) and $x_2^* = 0$ (unstable).

Stability can be determined from the slope of f at the fixed points: negative slope implies stable while positive slope implies unstable. Zero slope is semistable in this case.

(b) The critical value $r = 1$ results in a transcritical bifurcation.

(c) Use the Taylor expansion $\ln(1+x) \approx x - x^2/2$ to get

$$0 = f(x^*) = rx^* - \ln(1+x^*) \approx rx^* - x^* + \frac{(x^*)^2}{2} = x^*(r - 1 + \frac{1}{2}x^*)$$

to locate the second fixed point $x^* = 2(1-r) \neq 0$ for $r \neq 1$. See the plot in Fig.2 below for the bifurcation diagram depicting these fixed points.

NOTE: graphs were produced by wxMaxima, arranged using LibreOffice Draw, and exported to PDF for inclusion in this LaTeX file.

□

3. (Ex.3.4.3, p.91) Suppose that $x = x(t)$ and $\dot{x} = rx - 4x^3$.

(a) Sketch the qualitatively different phase diagrams as r is varied,

(b) Determine the critical value of r for which there is a pitchfork bifurcation.

(c) Sketch the bifurcation diagram of fixed points x^* versus r.

Solution: (a) See the graphs in Fig.3 below of the phase portrait $\dot{x} = f(x) = rx - 4x^3$. Three qualitatively different diagrams are:

- $r < 0$: One stable fixed point $x^* = 0$.
- $r = 0$: Semistable fixed point $x^* = 0$, in fact stable.
- $r > 0$: Three fixed points $x_1^* < 0$ (stable), $x_2^* = 0$ (unstable), and $x_3^* < 0$ (stable).

Stability can be determined from the slope of f at the fixed points: negative slope implies stable while positive slope implies unstable. Zero slope is semistable but further flow analysis shows stability in this case.

(b) The critical value $r = 0$ results in a supercritical pitchfork bifurcation.

(c) Factor the polynomial to get

$$0 = f(x^*) = rx^* - 4(x^*)^2 = x^*(r - 4(x^*)^2) = x^*(r - 4(x^*)^2) = 4x^*(\frac{\sqrt{r}}{2} - x^*)(\frac{\sqrt{r}}{2} + x^*)$$

to locate the nonzero fixed points $x^* = \pm\sqrt{r}/2 \neq 0$ for $r > 0$. See the plot in Fig.3 below for the bifurcation diagram depicting these fixed points.

NOTE: graphs were produced by wxMaxima, arranged using LibreOffice Draw, and exported to PDF for inclusion in this LaTeX file.

□

4. (Ex.3.4.[5,...,10], p.92) For the following dynamical systems, classify the type of bifurcations that arise: saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork.

(a) $\dot{x} = r - 3x^2$

(b) $\dot{x} = rx - \dfrac{x}{1+x}$

(c) $\dot{x} = 5 - re^{-x^2}$

(d) $\dot{x} = rx - \dfrac{x}{1+x^2}$

(e) $\dot{x} = x + \tanh(rx)$

(f) $\dot{x} = rx + \dfrac{x^3}{1+x^2}$

Solution: Use approximations and compare with standard forms.

(a) **Saddle Node**. Get standard form $\dot{u} = s + u^2$ after substitutions $x \leftarrow -u/3$ and $r \leftarrow -3s$.

(b) **Transcritical**. Use the geometric sum formula for small x to get

$$\dot{x} = rx - \frac{x}{1+x} = rx - x(1 - x + x^2 - x^3 + \cdots) \approx (r-1)x - x^2$$

Get standard form $\dot{u} = su - u^2$ after substitutions $x \leftarrow u$ and $r \leftarrow s + 1$.

(c) **Saddle Node**. Use Taylor's expansion for small x to get

$$\dot{x} = 5 - re^{-x^2} = 5 - r(1 - x^2 + \frac{1}{2}x^4 - \cdots) \approx (5 - r) - rx^2$$

Get standard form $\dot{u} = s + u^2$ after substitutions $x \leftarrow -u/r$ and $s \leftarrow r(r-5)$.

(d) **Subcritical Pitchfork**. Use the geometric sum formula for small x to get

$$\dot{x} = rx - \frac{x}{1+x^2} = rx - x(1 - x^2 + x^4 - \cdots) \approx (r-1)x + x^3$$

Get standard form $\dot{u} = su + u^3$ after substitutions $x \leftarrow u$ and $r \leftarrow s + 1$.

(e) **Supercritical Pitchfork**. Use Taylor's expansion of hyperbolic tangent $\tanh(rx) = \frac{1}{2}(e^{rx} + e^{-rx})$ for small x to get

$$\dot{x} = x + \tanh(rx) = x + (rx - \frac{(rx)^3}{3} + \cdots) \approx (r+1)x - r^3x^3$$

Get standard form $\dot{u} = su - u^3$ after substitutions $x \leftarrow u\sqrt{3/r^3}$ and $s \leftarrow r - 1$.

(f) **Subcritical Pitchfork**. Use the geometric sum formula for small x to get

$$\dot{x} = rx + \frac{x^3}{1+x^2} = rx + x^3(1 - x^2 + \cdots) \approx rx + x^3$$

This is the standard form $\dot{x} = rx + x^3$ without any substitutions.

□

5. (Ex.3.6.2, p.96) Consider the system $\dot{x} = h + rx - x^2$ for constants h, r. When $h = 0$ the system undergoes a transcritical bifurcation at $r = 0$.

(a) Plot the bifurcation diagrams for $h < 0$, $h = 0$, and $h > 0$.

(b) Sketch the regions in the (r, h)-plane that correspond to qualitatively different phase diagrams and identify the bifurcations that occur on the boundaries of these regions.

(c) Plot the potentials $V(x)$ corresponding to the different regions of part (b).

Solution: Begin by solving for the fixed points using the quadratic formula:

$$\dot{x} = 0 = h + rx - x^2 \iff 2x^* = r \pm \sqrt{r^2 + 4h}$$

Determine stability from the sign of $\frac{d}{dx}(h + rx - x^2) = r - 2x$

(a) There are three cases:

- $h = 0$: fixed points $x_1^* = 0$ and $x_2^* = r$, stable and unstable, transcritical bifurcation at $r = 0$.
- $h > 0$: two fixed points $x_1^* < 0$ and $x_2^* > 0$ for any r, stable and unstable, no bifurcations.
- $h < 0$: saddle node bifurcations at $r = \pm 2\sqrt{-h}$
  - no fixed points if $r^2 + 4h < 0$, since there are no real roots.
  - two fixed points x^* between 0 and r, if $r^2 + 4h > 0$.

See the bifurcation diagrams in Fig.4 (a).

(b) There are three qualitatively different regions in the (r, h) plane for this system. They are depicted in Fig.4 (b).

- $h > 0$: region above the red line (which depicts $h = 0$). Here there are no fixed points for any r.
- $h < 0$, $r^2 + 4h > 0$: region below the red line $h = 0$ and above the blue parabola ($h = -r^2/4$) where there are two fixed points.
- $h < 0$, $r^2 + 4h > 0$: region below the blue parabola ($h = -r^2/4$) where there are no fixed points.

The red line depicts transcritical bifurcations.

The blue parabola depicts saddle node bifurcations.

(c) Graphs of the potentials $V(x) = x^3 - rx^2/2 - hx$ are superimposed over their respective (r, h) regions in Fig.4 (c).

□

6. (Ex.4.1.[2,...,7], p.127) For each of the following dynamical systems, find and classify all fixed points and sketch the phase portrait on the circle.

(a) $\dot{\theta} = 1 + 2\cos\theta$

(b) $\dot{\theta} = \sin 2\theta$

(c) $\dot{\theta} = \sin^3\theta$

(d) $\dot{\theta} = \sin\theta + \cos\theta$

(e) $\dot{\theta} = 3 + \cos 2\theta$

(f) $\dot{\theta} = 3 + \sin k\theta$ where k is a positive integer.

Solution: For $\dot{\theta} = f(\theta)$, find fixed points by $f(\theta^*) = 0$ and classify stability by the sign of $f'(\theta^*)$.

(a) $f(\theta) = 1 + 2\cos\theta$, so fixed points are $\theta^* = \cos^{-1}(-1/2) = \pm 2\pi/3$.

Compute $f'(\theta) = -2\sin\theta$, so

- $f'(\theta^* = 2\pi/3) < 0$ implies **stable**
- $f'(\theta^* = -2\pi/3) > 0$ implies **unstable**

(b) $f(\theta) = \sin 2\theta$, so fixed points are $\theta^* = \frac{1}{2}\sin^{-1}(0) = 0, \pm\pi/2, \pi$.

Compute $f'(\theta) = 2\cos 2\theta$, so

- $f'(\theta^* = 0) > 0$ implies **unstable**
- $f'(\theta^* = \pi/2) < 0$ implies **stable**
- $f'(\theta^* = -\pi/2) < 0$ implies **stable**
- $f'(\theta^* = \pi) > 0$ implies **unstable**

(c) $f(\theta) = \sin^3\theta$, so fixed points are $\theta^* = \sin^{-1}(0) = 0, \pi$.

Compute $f'(\theta) = -3\sin^2\theta\cos\theta$, so

- $f'(\theta^* = 0) = 0$ implies *semistable*, though it is actually **unstable**.
- $f'(\theta^* = \pi) = 0$ implies *semistable*, though it is actually **stable**.

(d) $f(\theta) = \sin\theta + \cos\theta$, so fixed points are $\theta^* = \tan^{-1}(-1) = -\pi/4, 3\pi/4$.

Compute $f'(\theta) = \cos\theta - \sin\theta$, so

- $f'(\theta^* = -\pi/4) > 0$ implies **unstable**
- $f'(\theta^* = 3\pi/4) < 0$ implies **stable**

(e) $f(\theta) = 3 + \cos 2\theta \geq 2$ is never zero, so there are no fixed points. Flow is always counterclockwise, fastest near 0 and π, slowest near $\pm\pi/2$.

(f) $f(\theta) = 3 + \sin k\theta \geq 2$ is never zero, so there are no fixed points. Flow is always counterclockwise, fastest where $\sin kx = 1$ and slowest where $\sin kx = -1$. See the example with $k = 5$ in Fig.5.

NOTE: Flows on the circle and phase portraits over $[-\pi, \pi]$ are depicted in Fig.5 below. □

7. (Ex.4.3.1, p.129) The time to pass through a saddle-node bottleneck is

$$T = \int_{-\infty}^{\infty} \frac{dx}{r + x^2},$$

where $r > 0$ is small. Evaluate this integral with an appropriate trigonometric substitution.

Solution: For any $r > 0$, the substitution $x = u\sqrt{r}$ yields

$$T = \int_{-\infty}^{\infty} \frac{\sqrt{r}\,du}{r + ru^2} = \frac{1}{\sqrt{r}}\int_{-\infty}^{\infty} \frac{du}{1 + u^2} = \frac{1}{\sqrt{r}}\tan^{-1}u\Big|_{-\infty}^{\infty} = \frac{\pi}{\sqrt{r}},$$

so the bottleneck transit time is $\pi/\sqrt{r}$.

NOTE: use trigonometric substitution if you cannot remember the antiderivative of $1/(1 + x^2)$. □

8. (Ex.4.3.[3,...,8], pp.129–130) For each of the following flows on the circle, draw the phase portraits for qualitatively different values of the control parameter μ.

(a) $\dot{\theta} = \mu\sin\theta - \sin 2\theta$

(b) $\dot{\theta} = \dfrac{\sin\theta}{\mu + \cos\theta}$

(c) $\dot{\theta} = \mu + \cos\theta + \cos 2\theta$

(d) $\dot{\theta} = \mu + \sin\theta + \cos 2\theta$

(e) $\dot{\theta} = \dfrac{\sin\theta}{\mu + \sin\theta}$

(f) $\dot{\theta} = \dfrac{\sin 2\theta}{1 + \mu\sin\theta}$

Solution: See Fig.6 below for the phase portrait sketches. It is easier to plot these over the interval $\theta \in [-\pi, \pi]$ than on a unit circle.

(a) There are three qualitatively different ranges of μ:

- $\mu < -2$: stable $\theta_1^* = 0$, unstable $\theta_2^* = \pi$.
- $-2 < \mu < 2$: stable $\theta_1^* = 0$ and $\theta_2^* = \pi$, unstable $\theta_3^* \in (0, \pi)$ and $\theta_4^* \in (-\pi, 0)$.
- $\mu > 2$: unstable $\theta_1^* = 0$, stable $\theta_2^* = \pi$.

(b) There are three qualitatively different ranges of μ:

- $\mu < -1$: stable $\theta_1^* = 0$, unstable $\theta_2^* = \pi$.
- $-1 < \mu < 1$: unstable $\theta_1^* = 0$ and $\theta_2^* = \pi$. Solution fails to exist near $\cos\theta = -\mu$.
- $\mu > 1$: unstable $\theta_1^* = 0$, stable $\theta_2^* = \pi$.

(c) Observe that $-\frac{9}{8} \leq \cos\theta + \cos 2\theta \leq 2$, using calculus and double angle identities. Thus -2 and $\frac{9}{8}$, but also 0, are critical values separating four quantitatively different ranges of μ:

- $\mu < -2$: no fixed points. Flow is negative (clockwise) at a periodically varying rate.
- $-2 < \mu < 0$: unstable $\theta_1^* < 0$, stable $\theta_2^* > 0$.
- $0 < \mu < 9/8$: stable θ_1^*, θ_3^*, unstable θ_2^*, θ_4^*, with $-\pi < \theta_1^* < \theta_2^* < 0 < \theta_3^* < \theta_4^* < \pi$.
- $\mu > 9/8$: no fixed points. Flow is positive (counterclockwise) at a periodically varying rate.

(d) Observe that $-2 \leq \sin\theta + \cos 2\theta \leq \frac{9}{8}$, using calculus and double angle identities. Thus -2 and $\frac{9}{8}$, but also 0, are critical values separating four quantitatively different ranges of μ:

- $\mu < -9/8$: no fixed points. Flow is negative (clockwise) at a periodically varying rate.
- $-9/8 < \mu < 0$: stable θ_1^*, θ_3^*, unstable θ_2^*, θ_4^*, with $-\pi < \theta_1^* < \theta_2^* < 0 < \theta_3^* < \theta_4^* < \pi$.
- $0 < \mu < 2$: stable $\theta_1^* < 0$, unstable $\theta_2^* > 0$, with $-\pi < \theta_1^* < \theta_2^* < 0$.
- $\mu > 2$: no fixed points. Flow is positive (counterclockwise) at a periodically varying rate.

(e) -1, 0, and 1 are critical values separating four quantitatively different ranges of μ:

- $\mu < -1$: stable $\theta_1^* = 0$, unstable $\theta_2^* = \pi$.
- $-1 < \mu < 0$: stable $\theta_1^* = 0$, unstable $\theta_2^* = \pi$, some flows blow up in finite time.
- $0 < \mu < 1$: unstable $\theta_1^* = 0$, stable $\theta_2^* = \pi$, some flows blow up in finite time.
- $\mu > 1$: unstable $\theta_1^* = 0$, stable $\theta_2^* = \pi$.

(f) There are three qualitatively different ranges of μ:

- $\mu < -1$: stable $\theta_1^* = -\pi/2$, unstable $\theta_2^* = 0$, $\theta_3^* = \pi/2$, and $\theta_4^* = \pi$. Some flows blow up in finite time.
- $-1 < \mu < 1$: unstable $\theta_1^* = 0$ and $\theta_2^* = \pi$, stable $\theta_3^* = -\pi/2$ and $\theta_4^* = \pi/2$.
- $\mu > 1$: stable $\theta_1^* = \pi/2$, unstable $\theta_2^* = 0$, $\theta_3^* = -\pi/2$, and $\theta_4^* = \pi$. Some flows blow up in finite time.

□

9. (Ex.4.4.1, p.130) Find the conditions on the parameters b, m, g, L for which it is valid to approximate

$$mL^2\ddot{\theta} + bL^2\dot{\theta} + mgL\sin\theta = \Gamma$$

by its overdamped limit $bL^2\dot{\theta} + mgL\sin\theta = \Gamma$. Justify your answer.

Solution: Follow the reasoning in section 4.4, pages 115–117. First divide by mgL so that all summed terms are dimensionless (like $\sin\theta$):

$$\frac{L}{g}\ddot{\theta} + \frac{bL}{mg}\dot{\theta} + \sin\theta = \frac{\Gamma}{mgL}$$

Next, introducing a characteristic time $T > 0$ and substitute dimensionless $\tau \leftarrow t/T$, then differentiate with respect to τ using the chain rule:

$$\dot{\theta} \stackrel{\text{def}}{=} \frac{d\theta}{dt} = \frac{d\theta}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\theta}{d\tau} = \frac{1}{T}\theta',$$

where $\theta' \stackrel{\text{def}}{=} d\theta/d\tau$. Then rewrite the equation in terms of θ' and θ'' to get:

$$\frac{L}{gT^2}\theta'' + \frac{bL}{mgT}\theta' + \sin\theta = \frac{\Gamma}{mgL}$$

Finally, choose T such that the coefficient of θ' is 1:

$$\frac{bL}{mgT} = 1 \quad \Rightarrow T = \frac{bL}{mg}, \quad \Rightarrow T^2 = \frac{b^2L^2}{m^2g^2}$$

and apply this to the assumption that the coefficient of θ'' is negligibly small:

$$\frac{L}{gT^2} \ll 1 \quad \Longleftrightarrow \quad \frac{m^2 g}{b^2 L} \ll 1 \quad \Longleftrightarrow \quad b^2 \gg \frac{m^2 g}{L}$$

so $b^2 \gg m^2 g/L$ justifies the reduction of order.

NOTE: there is an error in the book's solution on p.541. This is evident from the similar analysis of the bead on a wire in Sec.3.5, Eq.4, p.73. □

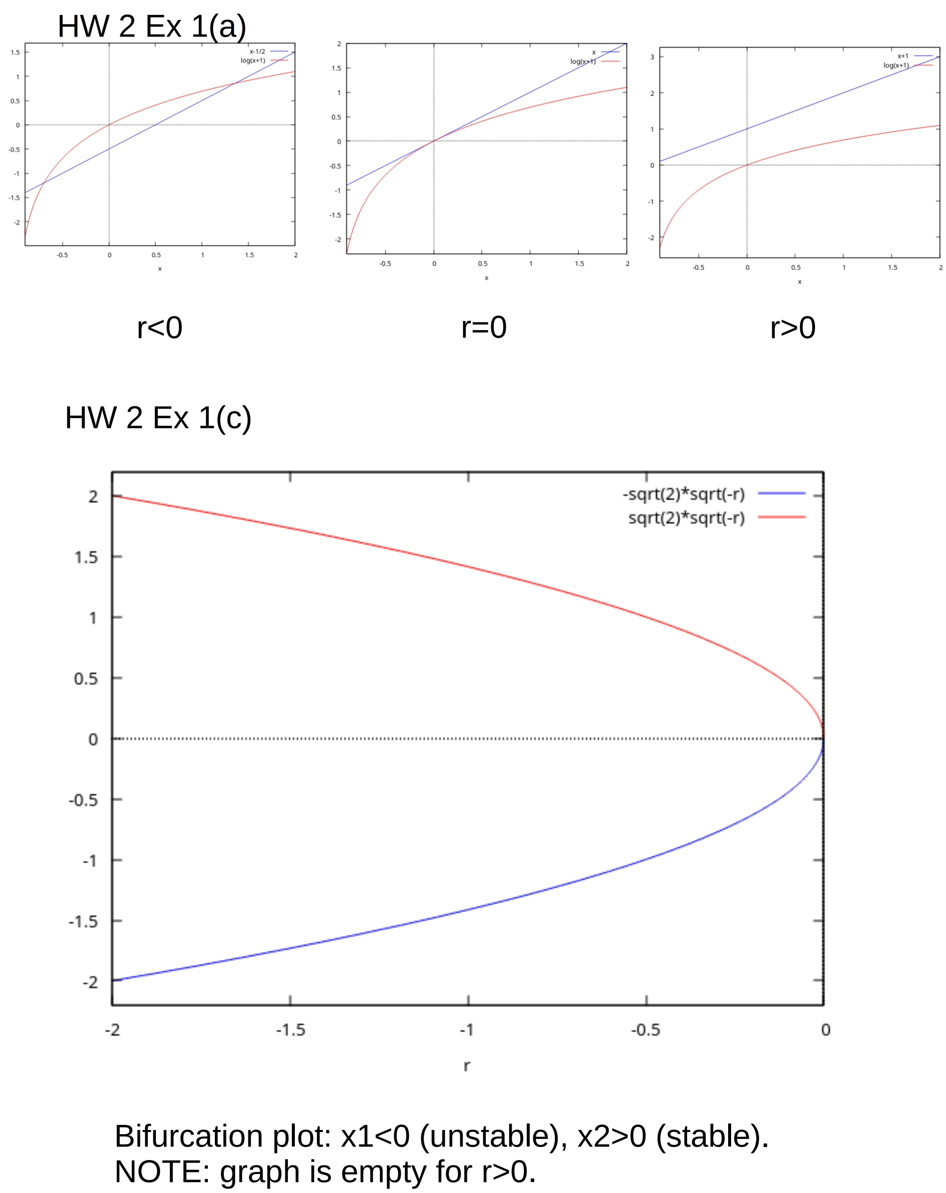


Figure 1: Phase and bifurcation diagrams for Exercise 1.

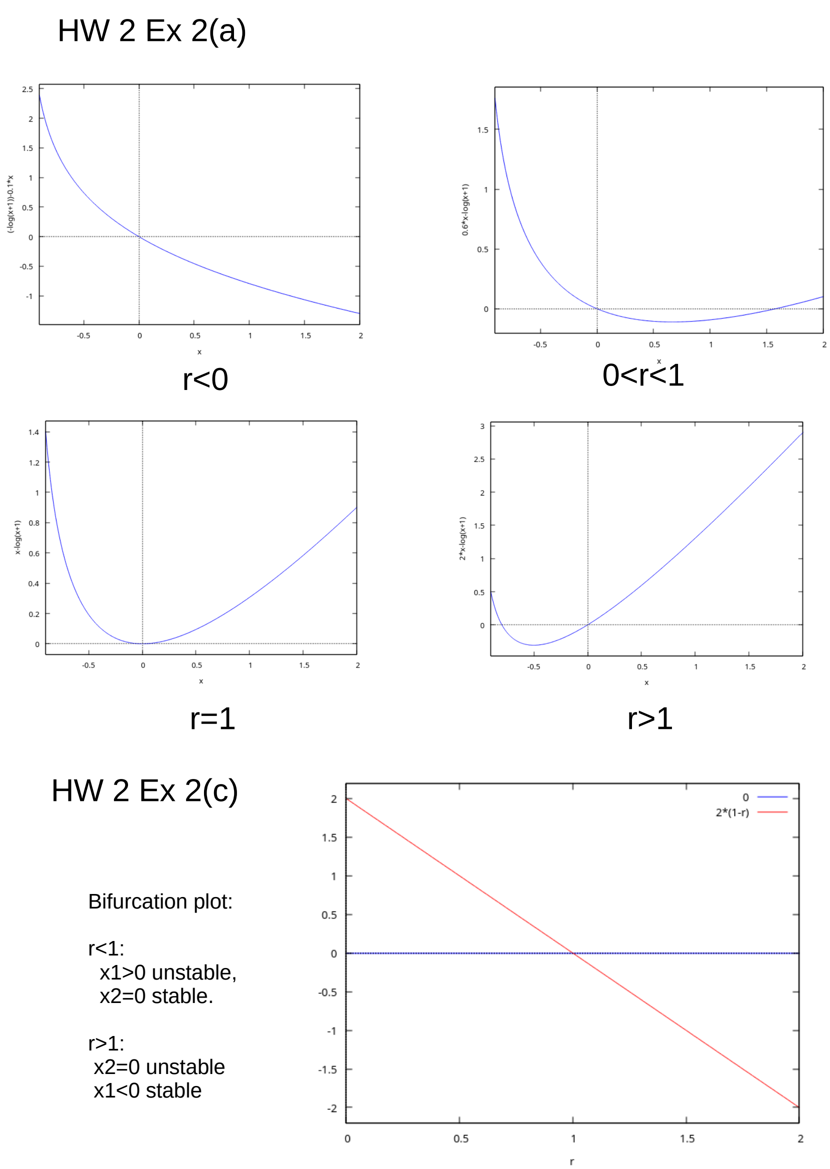
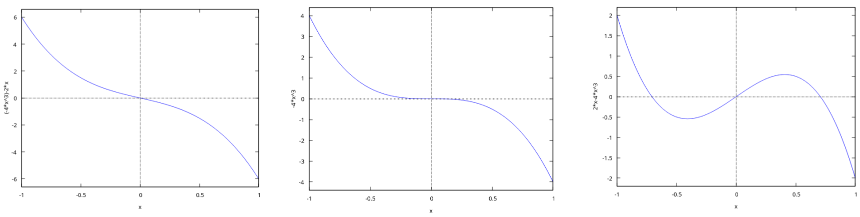


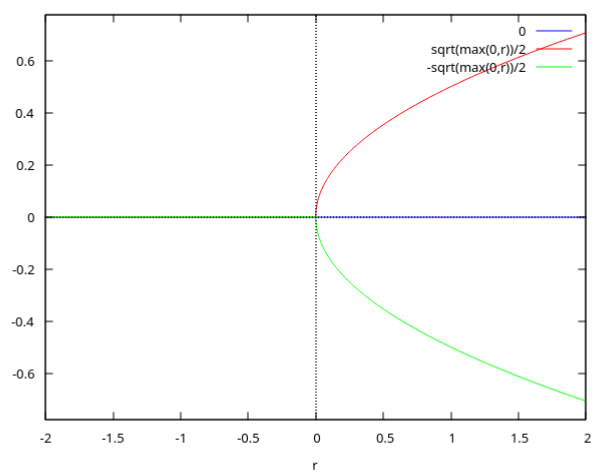
Figure 2: Phase and bifurcation diagrams for Exercise 2.

HW 2 Ex 3(a)



r<0 r=0 r>0

HW 2 Ex 3(c)



Bifurcation plot: x1<0 (stable), x3>0 (stable), and x2=0 (stable for r<0 or r=0, unstable for r>0).

Figure 3: Phase and bifurcation diagrams for Exercise 3.

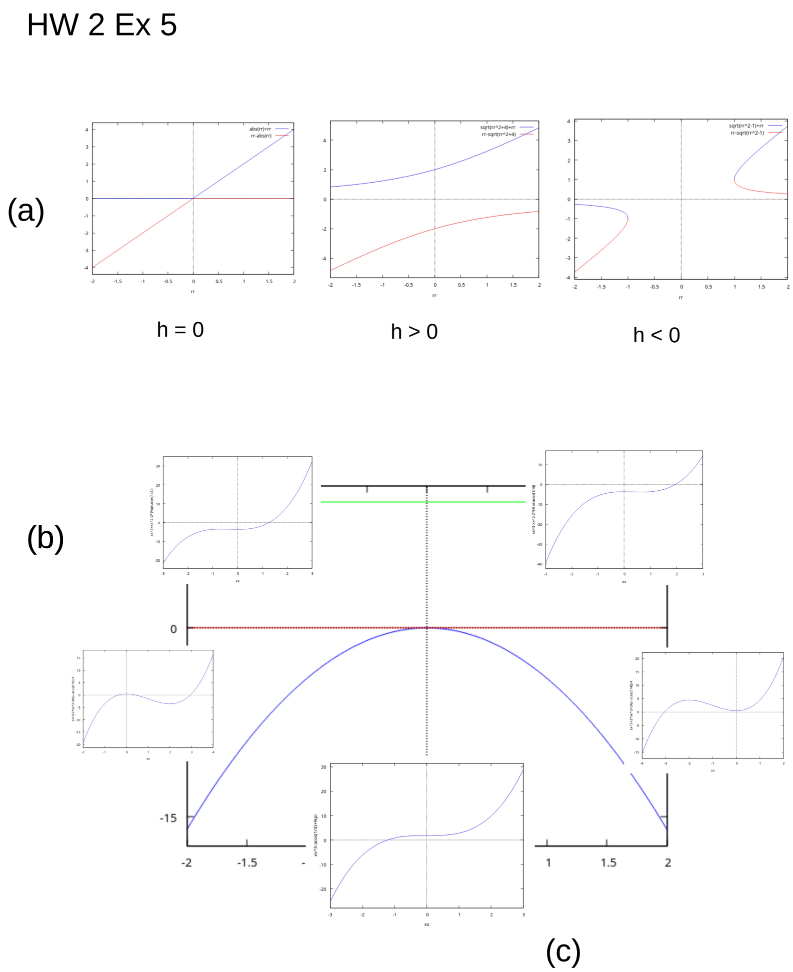


Figure 4: Bifurcation diagrams, (r,h) regions, and potentials for Exercise 5.

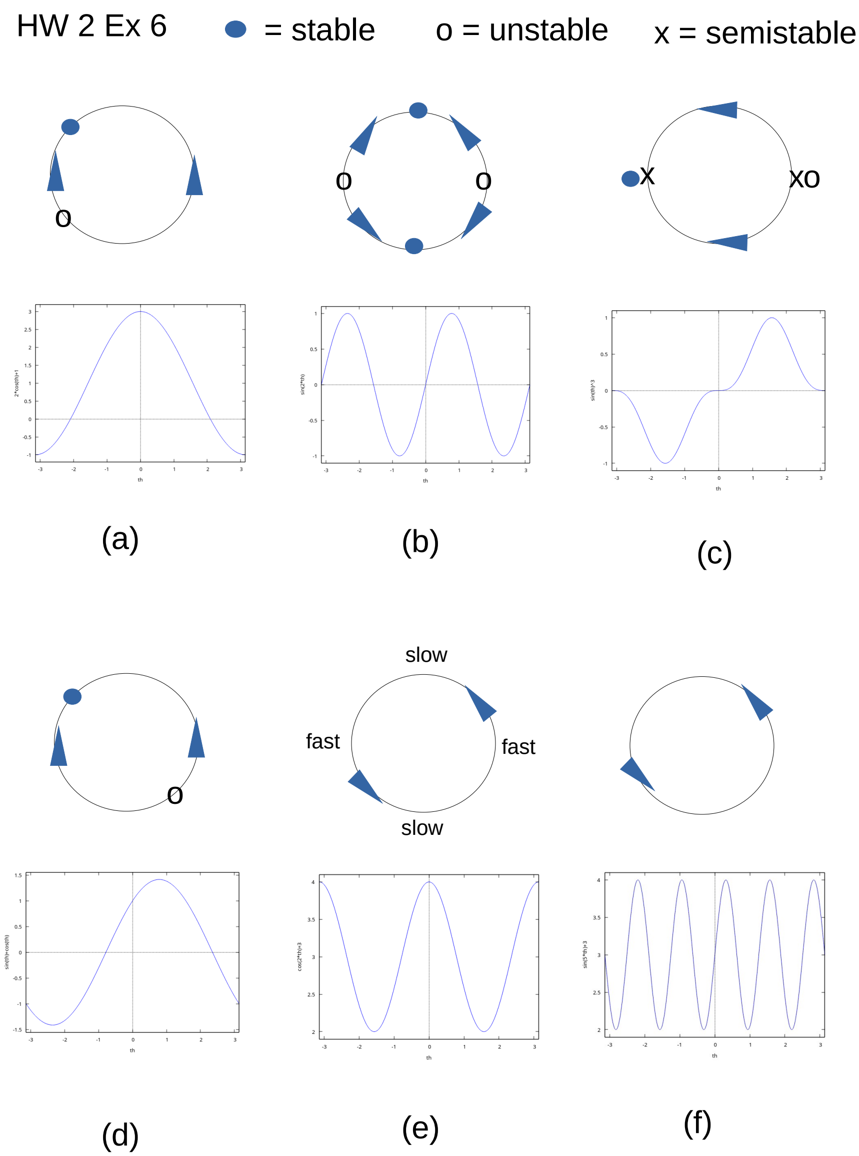


Figure 5: Phase diagrams and flows on the circle for Exercise 6.

HW 2 Ex 8

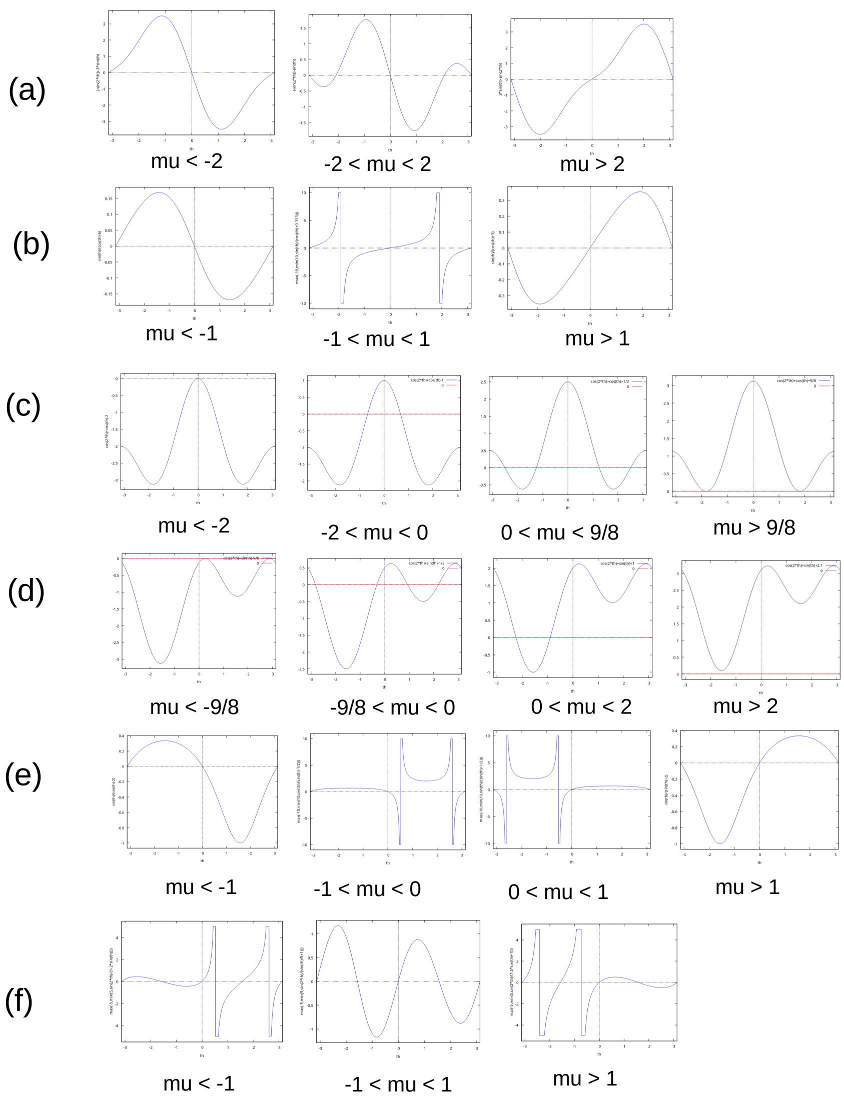


Figure 6: Phase diagrams for Exercise 8.