

Ma 3520: Differential Equations and Dynamical Systems

Solutions to Homework Assignment 3

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Read Chapters 5 and 6 of the textbook, “Nonlinear Dynamics and Chaos,” 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

1. (Ex.5.1.[3,4,5,6], p.155) Write the following linear systems in matrix form:

(a) $\dot{x} = -y, \dot{y} = -x.$

(b) $\dot{x} = 3x - 2y, \dot{y} = 2y - x.$

(c) $\dot{x} = 0, \dot{y} = x + y.$

(d) $\dot{x} = y, \dot{y} = 5x + y.$

Solution: These will all be in the form $\dot{X} = AX$ with row vector $X = (x, y)^T$, derivative $\dot{X} = (\dot{x}, \dot{y})^T$, and corresponding matrix A as below:

(a)

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

(d)

$$A = \begin{pmatrix} 0 & 1 \\ 5 & 1 \end{pmatrix}$$

□

2. (Ex.5.1.10, p.155) Read the definitions in the textbook’s preamble to this exercise. Then for each of the following systems, decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

(a) $\dot{x} = y, \dot{y} = -4x.$

(b) $\dot{x} = 2y, \dot{y} = x.$

(c) $\dot{x} = 0, \dot{y} = x.$

(d) $\dot{x} = 0, \dot{y} = -y.$

(e) $\dot{x} = -x, \dot{y} = -5y.$

(f) $\dot{x} = x, \dot{y} = y.$

Solution: The origin is evidently a fixed point for all of these linear autonomous systems $\dot{X} = AX$. Determine stability by finding eigenvalues of the matrix A , namely the roots of the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$.

(a)

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}; \quad p(\lambda) = \lambda^2 + 4; \quad \lambda = \pm 2i,$$

purely imaginary eigenvalues, so the origin is **Liapunov stable**.

(b)

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}; \quad p(\lambda) = \lambda^2 - 2; \quad \lambda = \pm\sqrt{2},$$

real eigenvalues of alternating sign, so the origin is a saddle which is **none of the above**.

(c)

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad p(\lambda) = \lambda^2; \quad \lambda = 0, 0,$$

so the system is degenerate. It is easy to see that any point $(x, 0)$ is a nonisolated repeller, so the origin is **none of the above**.

(d)

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}; \quad p(\lambda) = \lambda^2; \quad \lambda = 0, 0,$$

so the system is degenerate. It is easy to see that any point $(x, 0)$ is a nonisolated attractor, so the origin is **Liapunov stable**.

(e)

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}; \quad p(\lambda) = (\lambda + 1)(\lambda + 5); \quad \lambda = -1, -5,$$

strictly negative real eigenvalues, so the origin is an **attractor**.

(f)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad p(\lambda) = (\lambda - 1)^2; \quad \lambda = 1, 1,$$

strictly positive real eigenvalues, two linearly independent eigenvectors, so the origin is a repeller that is **none of the above**.

□

3. (Ex.5.2.[3,...,10], p.91) Plot the flow field in phase space and classify the fixed points of the following systems. If there are real eigenvectors, plot them on the diagram.

[Hint: see <https://www.math.wustl.edu/~victor/classes/ma3520/lin2d.txt>]

(a) $\dot{x} = y, \dot{y} = -2x - 3y.$

(b) $\dot{x} = 5x + 10y, \dot{y} = -x - y.$

(c) $\dot{x} = 3x - 4y, \dot{y} = x - y.$

(d) $\dot{x} = -3x + 2y, \dot{y} = x - 2y.$

(e) $\dot{x} = 5x + 2y, \dot{y} = -17x - 5y.$

(f) $\dot{x} = -3x + 4y, \dot{y} = -2x + 3y.$

(g) $\dot{x} = 4x - 3y, \dot{y} = 8x - 6y.$

(h) $\dot{x} = y, \dot{y} = -x - 2y.$

Solution: Eigenvalues are given and used to classify the fixed point at the origin.

See the phase portraits in Figure 1 below, generated by Octave.

(a) $\dot{x} = y, \dot{y} = -2x - 3y.$ Eigenvalues $0.56, -3.56$ imply **saddle**.

(b) $\dot{x} = 5x + 10y, \dot{y} = -x - y.$ Eigenvalues $2 \pm i$ imply **expanding spiral**.

(c) $\dot{x} = 3x - 4y, \dot{y} = x - y.$ Eigenvalues $1, 1$ with a single eigenvector imply **degenerate repeller**.

(d) $\dot{x} = -3x + 2y, \dot{y} = x - 2y.$ Eigenvalues $3.37, -2.37$ imply **saddle**.

(e) $\dot{x} = 5x + 2y, \dot{y} = -17x - 5y.$ Eigenvalues $\pm 3i$ imply **center**.

(f) $\dot{x} = -3x + 4y, \dot{y} = -2x + 3y.$ Eigenvalues ± 1 imply **saddle**.

(g) $\dot{x} = 4x - 3y, \dot{y} = 8x - 6y.$ Eigenvalues $-2, 0$ imply **nonisolated attractors**.

(h) $\dot{x} = y, \dot{y} = -x - 2y.$ Eigenvalues $-1, -1$ with a single eigenvector imply **degenerate attractor**.

□

4. (Ex.6.1.[8,9,10,11], p.199) Plot the flow fields in phase space for the following 2×2 dynamical systems:

[Hint: see <https://www.math.wustl.edu/~victor/classes/ma3520/rk4a.txt>]

(a) (Van der Pol's oscillator) $\dot{x} = y, \dot{y} = -x + y(1 - x^2).$

(b) (Dipole fixed point) $\dot{x} = 2xy, \dot{y} = y^2 - x^2.$

(c) (Two-eyed monster) $\dot{x} = y + y^2, \dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2.$

(d) (Parrot) $\dot{x} = y + y^2, \dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2.$

Solution: See the phase portraits in Figure 2 below, generated by Octave. □

5. (Ex.6.3.10, p.201) Consider the system $\dot{x} = xy, \dot{y} = x^2 - y$, for which linearization is inconclusive.

(a) Show that the linearization predicts that the origin is a non-isolated fixed point.

(b) Show that the origin is in fact an isolated fixed point.

(c) Classify the origin as attracting, repelling, saddle, or something else, using a sketch of the flow field along the nullclines.

(d) Plot a computer-generated flow field phase portrait to confirm your answer to part (c).

Solution: (a) Linearization $\dot{X} = AX$ at the origin has matrix $A = (y, x; 2x, -1)|_{(0,0)} = (0, 0; 0, -1)$ which has eigenvalues $0, -1$ with eigenvectors $(1; 0)$ and $(0; 1)$. Hence the origin is a non-isolated fixed point: all points of the form $(x, 0)$ are attracting.

(b) Including the nonlinear terms shows that for small $(x, y) \neq (0, 0)$,

- $y < 0$ implies $\dot{y} \approx -y, \dot{x} = yx$, which tends to $(0, 0)$.
- $y > 0$ implies x moves away from 0, so eventually (when $x^2 > y$) y stays positive and grows larger.

Conclude that $(x, 0)$ is not a fixed point for $x \neq 0$ for the nonlinear equation.

(c) From the mixture of attracting and repelling, classify $(0, 0)$ as an isolated **saddle**.

(d) See the phase portrait in Figure 3 below, generated by Octave. □

6. (Ex.6.4.[1,2,3], p.203) For each of the following “rabbits vs. sheep” dynamical systems, where $x, y \geq 0$, find the fixed points, classify them by stability, draw the nullclines, and indicate the basins of attraction for any stable fixed points.

(a) $\dot{x} = x(3 - x - y)$, $\dot{y} = y(2 - x - y)$.

(b) $\dot{x} = x(3 - 2x - y)$, $\dot{y} = y(2 - x - y)$.

(c) $\dot{x} = x(3 - 2x - 2y)$, $\dot{y} = y(2 - x - y)$.

Solution: Use Octave to compute the eigenvalues after finding the nullclines, the fixed points, and the Jacobian matrix by hand.

See Figure 4 below for the flow fields and nullclines.

(a) Nullclines $x = 0$, $y = 0$, $y = 3 - x$, and $y = 2 - x$. Jacobian matrix

$$J = \begin{pmatrix} 3 - 2x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}$$

Fixed points (at the intersections of the nullclines):

- $(0, 0)$, $\lambda = 2, 3$, unstable, repelling.
- $(0, 2)$, $\lambda = -2, 1$, unstable, saddle.
- $(3, 0)$, $\lambda = -3, 1$, stable, attracting.

The entire first quadrant is a basin of attraction to $(3, 0)$.

(b) Nullclines $x = 0$, $y = 0$, $y = 3 - 2x$, and $y = 2 - x$. Jacobian matrix

$$J = \begin{pmatrix} 3 - 4x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}$$

Fixed points (at the intersections of the nullclines):

- $(0, 0)$, $\lambda = 2, 3$, unstable, repelling.
- $(0, 2)$, $\lambda = -2, 1$, unstable, saddle.
- $(3/2, 0)$, $\lambda = -3, 1/2$, unstable, saddle.
- $(1, 1)$, $\lambda = -2.6, -0.4$, stable, attracting.

The entire first quadrant is a basin of attraction to $(1, 1)$.

(c) $\dot{x} = x(3 - 2x - 2y)$, $\dot{y} = y(2 - x - y)$. Nullclines $x = 0$, $y = 0$, $y = 3/2 - x$, and $y = 2 - x$. Jacobian matrix

$$J = \begin{pmatrix} 3 - 4x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}$$

Fixed points (at the intersections of the nullclines):

- $(0, 0)$, $\lambda = 2, 3$, unstable, repelling.

- $(0, 2)$, $\lambda = -2, -1$, stable, attracting.
- $(3/2, 0)$, $\lambda = -3, 1/2$, unstable, saddle.

The entire first quadrant is a basin of attraction to $(0, 2)$. □

7. (Ex.6.5.2, p.207) Consider the system $\ddot{x} = x - x^2$.

- Find and classify the equilibrium points by stability.
- Sketch the phase portrait flow field.
- Find an equation for the homoclinic orbit that separates closed and nonclosed trajectories.

Solution: Convert to a first-order system:

$$\dot{x} = y; \quad \dot{y} = x - x^2 = x(1 - x).$$

Jacobian matrix:

$$J = \begin{pmatrix} 0 & 1 \\ 1 - 2x & 0 \end{pmatrix}$$

Note that the original nonlinear system is both reversible and conservative, with potential function V and total energy function E given by

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{3}x^3, \quad E(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3,$$

where conserved quantity E is obtained from the exact equation

$$0 = \dot{x}(\ddot{x} + x - x^2) = \frac{d}{dt} \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 \right] = \frac{d}{dt} \left[\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 \right]$$

(a) Equilibrium points $(\dot{x}, \dot{y}) = (0, 0)$ with the eigenvalues of their linearizations are

- $(0, 0)$, $\lambda = \pm 1$, stable, saddle.
- $(1, 0)$, $\lambda = \pm i$, linear center, in a reversible system so it is a stable nonlinear center.

(b) See the phase portrait in Figure 3 below, generated by Octave.

(c) The homoclinic orbit connecting the saddle $(0, 0)$ to itself has constant total energy $E(0, 0) = 0$, so its equation is

$$y^2 = x^2 - \frac{2}{3}x^3, \quad \Rightarrow y = \pm \sqrt{x^2 - \frac{2}{3}x^3}.$$

This may be plotted in two parts, over the range $0 \leq x \leq 3/2$, as seen in (b) graph. □

8. (Ex.6.5.4, p.207) Sketch the phase portrait for the system $\ddot{x} = ax - x^2$ for $a < 0$, $a = 0$, and $a > 0$.

Solution: Convert to a first-order system:

$$\dot{x} = y; \quad \dot{y} = ax - x^2 = x(a - x).$$

Jacobian matrix:

$$J = \begin{pmatrix} 0 & 1 \\ a - 2x & 0 \end{pmatrix}$$

See the phase portraits in Figure 5 below, generated by Octave, corresponding to $a = -2$, $a = 0$, and $a = +2$. □

9. (Ex.6.6.2, p.212) Show that the system $\dot{x} = y$, $\dot{y} = x \cos y$ is reversible, and sketch its phase portrait.

Solution: Since $y \mapsto y$ is an odd function while $y \mapsto \cos y$ is an even function, the system is reversible under the involution $R(x, y) = (x, -y)$. See its phase portrait in Figure 5 below, generated by Octave, \square

10. (Ex.6.6.10, p.214) Is the origin a nonlinear center for the system $\dot{x} = -y - x^2$, $\dot{y} = x$?

Solution: Yes. It is a linear center since the Jacobian matrix is

$$J(x, y) = \begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix}, \quad \Rightarrow J(0, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so the linearization at $(0, 0)$ has eigenvalues $\pm i$, indicating a center.

The original nonlinear system is reversible under the involution $R(x, y) = (x, -y)$, so the linear center is a nonlinear center. \square

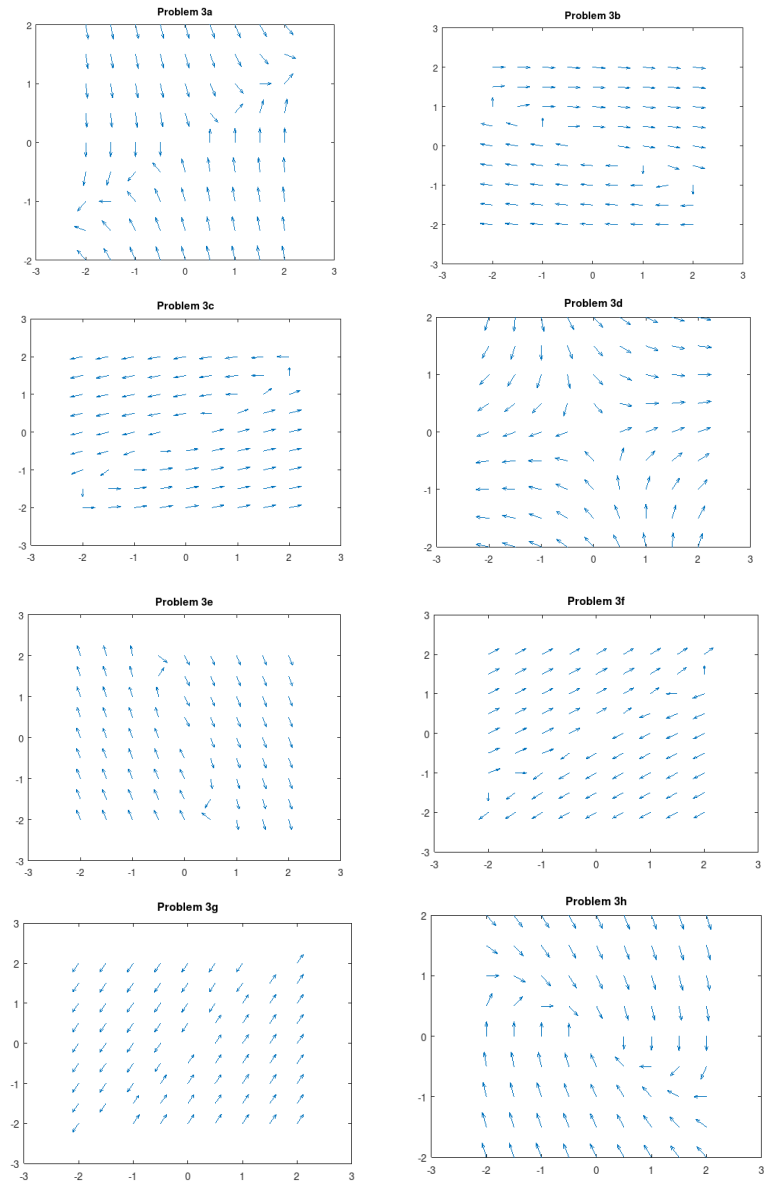


Figure 1: Phase diagrams for Exercise 3.

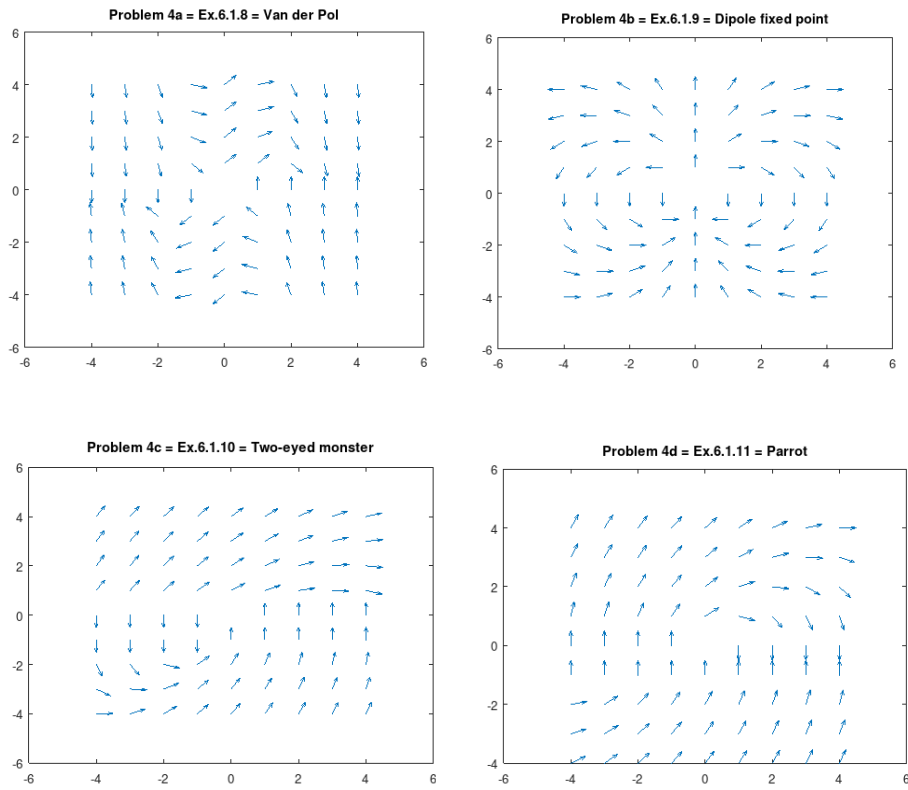


Figure 2: Phase diagrams for Exercise 4.

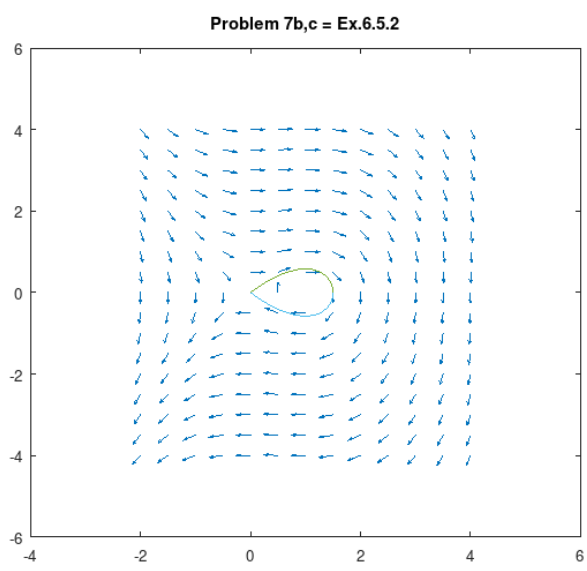
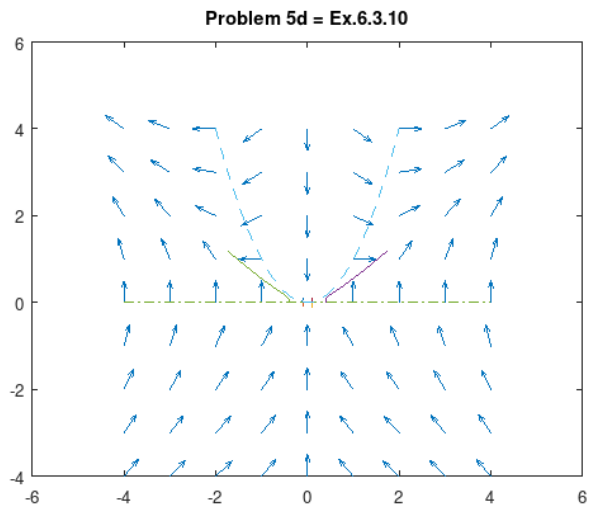


Figure 3: Phase diagrams for Exercises 5 and 7.

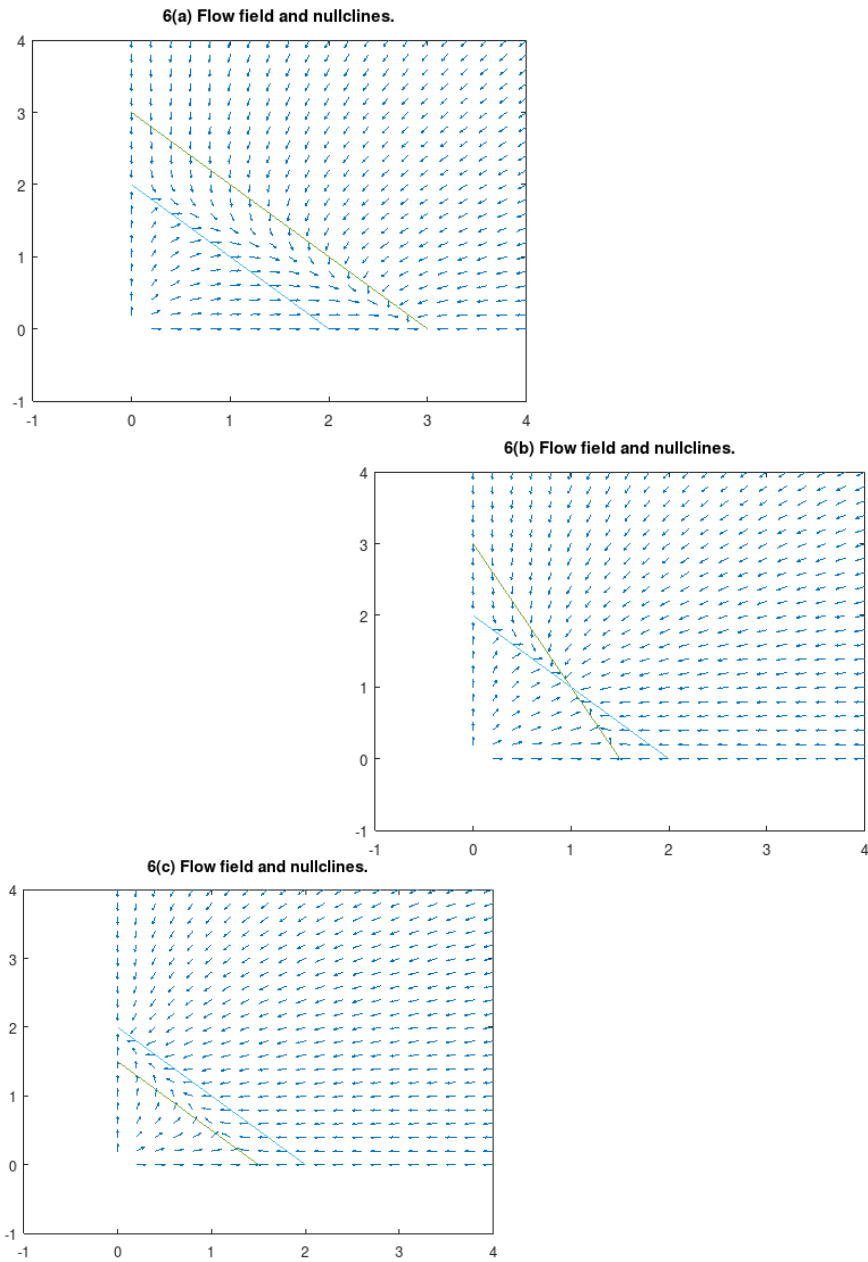


Figure 4: Flow fields and nullclines for Exercise 6.

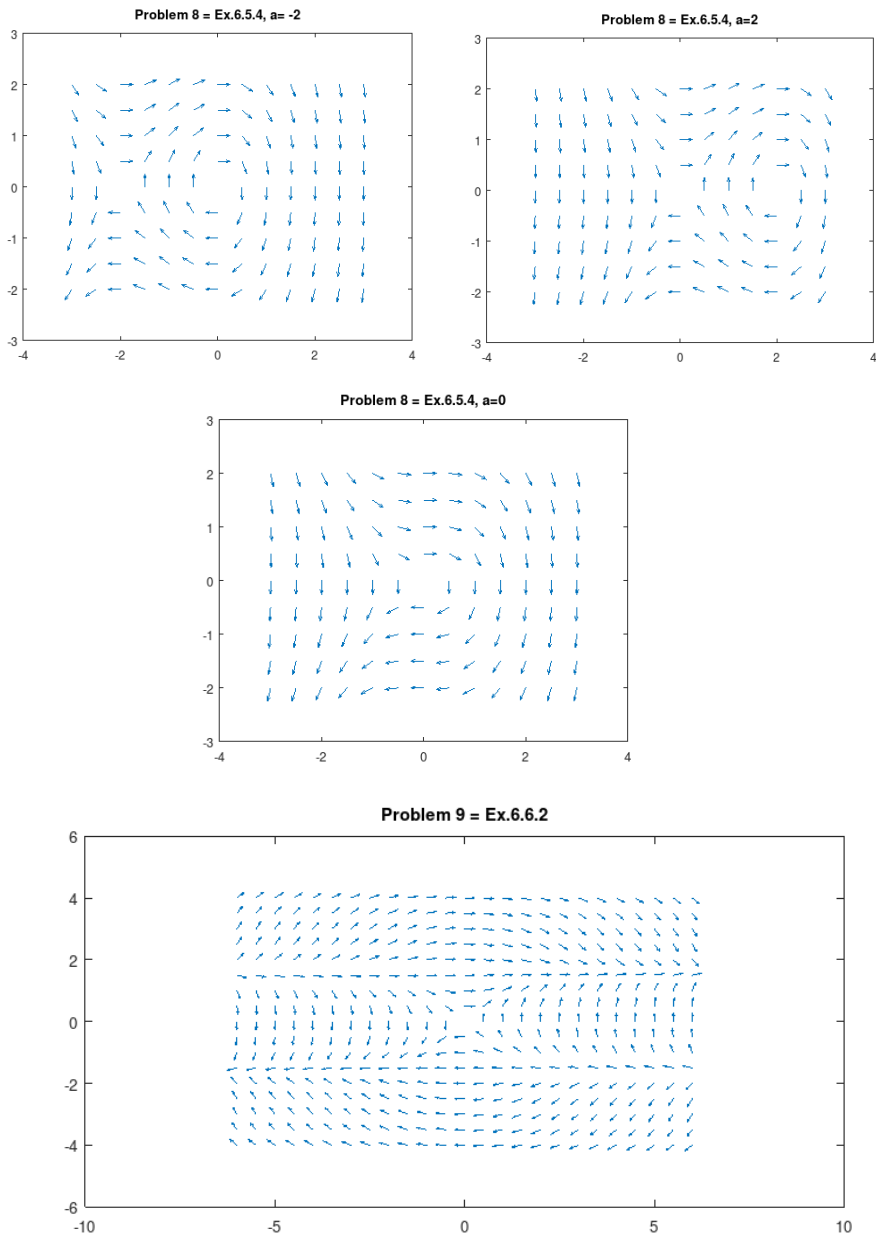


Figure 5: Phase diagrams for Exercises 8 and 9.