Ma 3520: Differential Equations and Dynamical Systems
Solutions to Homework Assignment 4

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Read Chapter 7 and 8 of the textbook, "Nonlinear Dynamics and Chaos," 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

1. (Ex.7.1.8abc, p.251) Consider the nonlinear oscillator whose governing equation is

$$\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0, \qquad a > 0.$$

(a) Find and classify all the fixed points.

(b) Show that the system has a circular limit cycle, and find its amplitude.

(c) Determine the stability of the limit cycle.

Solution: (a) Convert to a first-order system $\dot{x} = y$, $\dot{y} = -ay(x^2 + y^2 - 1) - x$. Then the unique fixed point is evidently $(x^*, y^*) = (0, 0)$, the origin.

Classify it using linearization with the Jacobian matrix

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -2axy - 1 & -a(x^2 + 3y^2 - 1) \end{pmatrix}, \quad \Rightarrow J(x^*, y^*) = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix},$$

which has eigenvalues $\lambda = \frac{1}{2}[a \pm \sqrt{a^2 - 4}]$. There are 3 cases:

- $0 < a < 2$, giving two complex eigenvalues with positive real part, a repelling linear center,
- $a = 2$, giving repeated real eigenvalue $a/2 > 0$, a repeller,
- $a > 2$, giving distinct positive real eigenvalues, also a repeller.

(b) Convert the first-order system to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ to get

$$\begin{aligned} \frac{d}{dt}(r\cos\theta) &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta = r\sin\theta, \\ \frac{d}{dt}(r\sin\theta) &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -ar\sin\theta(r^2 - 1) - r\cos\theta, \end{aligned}$$

after the simplification $r^2 = x^2 + y^2$. Multiply the top equation by $\cos\theta$ and the bottom by $\sin\theta$, then add to get

$$\dot{r} = ar(1 - r^2)\sin^2\theta.$$

Evidently there is a circular orbit with $r = 1$.

(c) Since $ar\sin^2\theta > 0$ unless $\theta = k\pi$ for $k \in \mathbf{Z}$, note that

- $0 < r < 1$ implies $\dot{r} > 0$, while
- $1 < r$ implies $\dot{r} < 0$.

Thus the circular orbit $r = 1$ is attracting, hence stable.

The circular limit cycle is in fact stable for all $a > 0$ since the origin is repelling in all cases. □

2. (Ex.7.2.7, p.253) Consider the system $\dot{x} = y + 2xy \overset{\text{def}}{=} f(x, y)$, $\dot{y} = x + x^2 - y^2 \overset{\text{def}}{=} g(x, y)$.

(a) Show that $\partial f/\partial y = \partial g/\partial x$, so that it is a gradient system.

(b) Find a potential V such that $(\dot{x}, \dot{y}) = -\nabla V(x, y)$. [Hint: use partial integration as in the solution of exact differential equations.]

(c) Sketch the system's phase portrait, and include the level curves of V.

Solution: (a) Compute the mixed partials:

$$\frac{\partial f}{\partial y} = 1 + 2x; \qquad \frac{\partial g}{\partial x} = 1 + 2x.$$

Hence there exists a potential function $V = V(x, y)$ satisfying $f = -\partial V/\partial x$ and $g = -\partial V/\partial y$.

(b) Find the x-antiderivative of f (up to an arbitrary function C_1 of just y) and the y-antiderivative of g (up to an arbitrary function C_2 of just x) and combine them:

$$\int f\,dx = xy + x^2y + C_1(y); \qquad \int g\,dy = xy + x^2y - \frac{1}{3}y^3 + C_2(x).$$

These are equal for the choices $C_2(x) = 0$ and $C_1(y) = -\frac{1}{3}y^3$, giving the potential

$$V(x, y) = \frac{1}{3}y^3 - xy - x^2y.$$

(c) Modify the Octave commands in `contour.txt` on the class website to obtain the graph in Figure 1 below. □

3. (Ex.7.2.10, p.254) Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

Solution: Find all fixed points at intersections of the nullclines $y = x^3$ and $x = -y^3 \iff y = -\sqrt[3]{x}$. The unique solution is $(x^*, y^*) = (0, 0)$.

So long as $a > 0$ and $b > 0$, we will have $V(x, y) > 0$ except at the unique fixed point $x^* = 0$, $y^* = 0$. Along trajectories,

$$\frac{d}{dt}V(x,y) = 2ax\dot{x} + 2by\dot{y} = 2ax(y - x^3) - 2by(x + y^3) = -2(ax^4 + by^4) + 2(a - b)xy,$$

which will be strictly negative away from the origin if we take positive $a = b$. Fix $a = b = 1$ to obtain a Liapunov function $V(x, y) = x^2 + y^2$, and conclude that the system has no closed orbits. □

4. (Ex.7.3.3, p.256) Use the Poincaré-Bendixson theorem to show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.

Solution: First find any fixed points at the intersections of the nullclines

$$x - y - x^3 = 0 \iff y = x - x^3; \qquad x + y - y^3 = 0 \iff x = y^3 - y.$$

Note that the rotation by $\pi/2$ given by $(x, y) \mapsto (y, -x)$ sends one these graphs into the other. Conclude that the origin is the only fixed point: $(x^*, y^*) = (0, 0)$.

Now find a trapping region that excludes the origin. A little experimentation finds that the square $[-3, 3] \times [-3, 3]$ is big enough to identify a trapping region

$$R = \{(x, y) : |x| \leq 2, |y| \leq 2\} \setminus \{(x, y) : |x + y| \leq \frac{1}{2}, |x - y| \leq \frac{1}{2}\},$$

namely the square of side 4 minus the small inner diamond, both centered at the origin. This is a trapping region because:

- At the outer boundary:
  - (Right) If $x = 2$ and $|y| \leq 2$, then $\dot{x} = -6 - y < 0$ points inward (left).
  - (Left) If $x = -2$ and $|y| \leq 2$, then $\dot{x} = 6 - y > 0$ points inward (right).
  - (Top) If $y = 2$ and $|x| \leq 2$, then $\dot{y} = x - 6 < 0$ points inward (down).
  - (Bottom) If $y = -2$ and $|x| \leq 2$, then $\dot{y} = x + 6 > 0$ points inward (up).
- At the inner boundary:
  - (NE) If $x + y = 1/2$ and $0 \leq x, y \leq 1/2$, then $\dot{y} = 1/2 - y^3 > 0$ points outward (up).
  - (SW) If $x + y = -1/2$ and $-1/2 \leq x, y \leq 0$, then $\dot{y} = -1/2 - y^3 < 0$ points outward (down).
  - (SE) If $x - y = 1/2$ and $0 \leq x \leq 1/2$ and $-1/2 \leq y \leq 0$, then $\dot{x} = 1/2 - x^3 > 0$ points outward (right).
  - (NW) If $x - y = -1/2$ and $-1/2 \leq x \leq 0$ and $0 \leq y \leq 1/2$, then $\dot{x} = -1/2 - x^3 < 0$ points outward (left).

Alternatively, use linearization at the fixed point $(0,0)$ to determine the flow behavior at the inner boundary of R. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} 1-3x^2 & -1 \\ 1 & 1-3y^2 \end{pmatrix}, \quad \Rightarrow J(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

so the (distinct, complex) eigenvalues $\lambda = 1 \pm i$ have positive real part, so the origin is repelling, so any sufficiently small exclusion region around $(0,0)$ will have only outflows across its boundary. This region may be used instead of the inner diamond above.

Conclude by the Poincaré-Bendixson theorem that there exists a periodic closed orbit within the domain R. See Figure 1 below for the phase portrait used here. □

5. (Ex.7.4.1, p.259) Use Liénard's theorem to show that the equation $\ddot{x} + \mu(x^2-1)\dot{x} + \tanh x = 0$, with $\mu > 0$ has exactly one periodic solution, and classify its stability.

Solution: The solution is merely to identify applicability and check the hypotheses.

First, rewrite the equation as a first-order system in Liénard's form

$$\dot{x} = y, \qquad \dot{y} = -\tanh x - \mu(x^2-1)y = -g(x) - f(x)y,$$

where

$$g(x) \stackrel{\text{def}}{=} \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad f(x) \stackrel{\text{def}}{=} \mu(1-x^2).$$

Second, note that f, g satisfy the five hypotheses of Liénard's theorem on p.234 of the textbook:

(1) f and g are continuously differentiable for all $x \in \mathbf{R}$;

(2) $g(-x) = -g(x)$ for all $x \in \mathbf{R}$;

(3) $x > 0 \Rightarrow g(x) > 0$;

(4) $f(x) = f(-x)$ for all $x \in \mathbf{R}$;

(5) The antiderivative $F(x) \stackrel{\text{def}}{=} \int_0^x f(u)\,du = \mu(\frac{1}{3}x^3 - x)$, which is an odd function, has a unique positive zero at $x = a \stackrel{\text{def}}{=} 1/\sqrt{3}$, is negative for $0 < x < a$, and is nondecreasing for $x > a$ with $F(x) \to \infty$ as $x \to \infty$.

Conclude that Liénard's theorem applies, so the system has a unique, stable limit cycle surrounding the origin in the phase plane. □

6. (Ex.8.2.1, p.315) Consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2-1)\dot{x} + x = a$. Find the curves in (μ, a)-space where Hopf bifurcations occur.

Solution: Rewrite the equation as a first-order system to identify the fixed point:

$$\dot{x} = y, \quad \dot{y} = -\mu(x^2-1)y - (x-a),$$

from which it is evident that the unique fixed point is $(x^*, y^*) = (a, 0)$.

Linearize around the fixed point and compute the Jacobian matrix:

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -2\mu xy - 1 & -\mu(x^2-1) \end{pmatrix}, \quad \Rightarrow J(a,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu(1-a^2) \end{pmatrix},$$

which has determinant $\Delta = +1$ and trace $\tau = \mu(1-a^2)$, so its eigenvalues are

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{\mu(1-a^2) \pm \sqrt{\mu^2(1-a^2)^2 - 4}}{2}.$$

Thus the linearized fixed point changes from a spiral (two complex conjugate eigenvalues) when $\mu^2(1-a^2)^2 < 4$, to a star (distinct real eigenvalues of the same sign) when $\mu^2(1-a^2)^2 > 4$, along the curves

$$\mu^2(1-a^2)^2 = 4, \quad \iff \mu_c = \pm\frac{2}{1-a^2}, \quad \iff a = \pm\sqrt{1 \pm \frac{2}{\mu_c}},$$

which has solutions for all $a \notin \{1, -1\}$. The Hopf bifurcations occur along those curves. They are plotted in Figure 2 below. It is easier to parameterize the curves in the a variable. The region with $|\mu| < |\mu_c|$ gives rise to cycles.

Note that if $a^2 > 1$, then both eigenvalues will have negative real part so the fixed point will be attracting and there will be no limit cycles. □

7. (Ex.8.2.6, p.316) In the system $\dot{x} = \mu x + y - x^3$, $\dot{y} = \mu y - x - 2y^3$, a Hopf bifurcation occurs at the origin when $\mu = 0$. Plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. [Hint: check your answer using the method of Ex.8.2.12, p.318.]

Solution: It is necessary to plot the phase portrait with various μ on either side of zero. This is done in Figure 2 below.

From these flows, it appears that $\mu = 0$ is a **supercritical** Hopf bifurcation. It separates a stable absorbing fixed point at the origin (for $\mu < 0$) from a stable limit cycle around the origin (for $\mu > 0$).

To check this with the method of Ex.8.2.12, note that at the Hopf bifurcation value $\mu_c = 0$, the system becomes

$$\begin{aligned} \dot{x} &= \mu_c x + y - x^3 = y - x^3 = -\omega y + f(x,y), \\ \dot{y} &= \mu_c y - x - 2y^3 = -x - 2y^3 = \omega x + g(x,y), \end{aligned}$$

where $\omega = -1$, $f(x,y) \stackrel{\text{def}}{=} -x^3$, and $g(x,y) \stackrel{\text{def}}{=} -2y^3$. Note that f and g are both higher-order nonlinear functions of x, y that vanish at the origin, as the criterion requires.

Compute the discriminant a as follows:

$$\begin{aligned} 16a &= f_{xxx} + f_{yyy} + g_{xxx} + g_{yyy} + \frac{1}{\omega}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} - f_{yy}g_{yy}\right] \\ &= -6 + 0 + 0 - 12 + (-1)\left[0 - 0 - (-6x)(0) - (0)(-12y)\right] = -18, \end{aligned}$$

from which we conclude that $a < 0$ so the Hopf bifurcation is **supercritical**. □

8. (Ex.8.4.1, p.322) For the system $\dot{r} = r(1 - r^2)$, $\dot{\theta} = \mu - \sin\theta$:

(a) Let $x = r\cos\theta$ and $y = r\sin\theta$, write the system in terms of x, y.

(b) For some μ slightly greater that $\mu_c = 1$, sketch the trajectories $x(t), y(t)$ for various initial $x(0), y(0)$. [Hint: use `rk4a()` from the class website.]

(c) Estimate, by trial and error with different μ, the relationship between $\mu - \mu_c$ and the period of a trajectory near the infinite-period bifurcation at μ_c.

Solution: (a) First note that $r^2 = x^2 + y^2$, and that $\sin\theta = y/\sqrt{x^2 + y^2}$.

Next, multiply $\dot{r} = r(1 - r^2)$ by $\cos\theta$ and $\sin\theta$, respectively, and use the product and chain rule formulas for the derivative with respect to t:

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}, \\ \dot{y} &= \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}. \end{aligned}$$

This gives two equations in just x, y, after eliminating $\dot{\theta} = \mu - \sin\theta = \mu - y/\sqrt{x^2 + y^2}$:

$$\begin{aligned} \dot{x} &= x(1 - x^2 - y^2) + y\dot{\theta} = x(1 - x^2 - y^2) + y\left(\mu - \frac{y}{\sqrt{x^2 + y^2}}\right), \\ \dot{y} &= y(1 - x^2 - y^2) - x\dot{\theta}. = y(1 - x^2 - y^2) - x\left(\mu - \frac{y}{\sqrt{x^2 + y^2}}\right). \end{aligned}$$

(b) At $\mu = \mu_c = 1$, there is a fixed point at $r = 1$, $\theta = \pi/2$, equivalently $(x, y) = (0, 1)$. This is the 12 o'clock position of the unit circle, and it is the starting and finishing point of the infinite-period unit circle trajectory.

At $\mu = 1.1$, which is slightly greater that $\mu_c = 1$, the trajectories starting from various initial points $(x(0), y(0))$ inside and outside the limit cycle are depicted in Figure 3 below.

(c) To estimate the period for $\mu > \mu_c$, start at $(x_0, y_0) = (0, 1)$ and count the number of time steps needed for one lap around the unit circle. This is done by trial and error with the `rk4a()` algorithm and `plot()`. For example, at $\mu = 1.1$, the two running times $T = 12.0$ and $T = 13.6$ produce the nearly complete orbits seen in Figure 3. Trying $T = 13.7$ filled the final gap, giving the estimate

$$13.6 < T(\mu = 1.1) < 13.7, \qquad \Rightarrow T(\mu = 1.1) \approx 13.65.$$

Similar efforts yielded the following table of values:

μ	$\mu-\mu_c$	$\frac{1}{\mu-\mu_c}$	$T(\mu)$, averaged	$T(\mu)^2$
1.5	0.5	2.0	5.55	30.8
1.3	0.3	3.3	7.55	57.0
1.1	0.1	10	13.65	186
1.05	0.05	20	19.6	384
1.02	0.02	50	31.5	992
1.01	0.01	100	44.5	1980

Plotting T versus $1/(\mu-\mu_c)$ yields a curve, but plotting T^2 versus $1/(\mu-\mu_c)$ yields a straight line as seen in Figure 3. Conclude that $T^2 = O(1(\mu-\mu_c))$, and so $T = O\left(1/\sqrt{\mu-\mu_c}\right)$ as $\mu \searrow \mu_c$. □

9. (Ex.8.4.2, p.322) What types of global bifurcations occur in the system $\dot{r} = r(\mu - \sin r)$, $\dot{\theta} = 1$ for various values of μ?

Solution: Because $\dot{\theta} = 1$, there are no fixed points, but there are cycles of various radii and stabilities. Find these by using the nullcline equation $\dot{r} = r(\mu - \sin r) = 0$ to relate μ and r.

The stability of the cycle of radius r_*, satisfying $\mu - \sin r_* = 0$, may be determined from the sign of the derivative with respect to r of the factor $\mu - \sin r$, namely the sign of $-\cos r_*$. The cycle is attracting for negative and repelling for positive as in the case of flows on the line.

If the derivative is zero then one side will be attracting while the other side will be repelling.

If there is no cycle then the origin will be a fixed point whose stability is determined by the sign of μ.

More precisely:

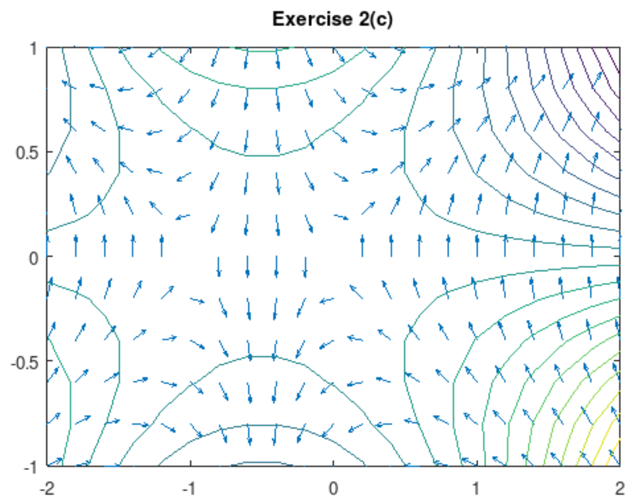
- $\mu > 1$: then $\mu - \sin r > 0$, all r, so the origin $r = 0$ is repelling and there are no cycles.
- $\mu = 1$: then $\dot{r} = 0 \iff r = \frac{\pi}{2} + 2k\pi$ for $k = 0, 1, 2, \ldots$, and all these radii are limit cycles attracting trajectories just inside themselves but repelling trajectories just outside.
- $0 < \mu < 1$: for each $k = 0, 1, 2, \ldots$, there are two radii $r_{k\pm} = 2k\pi + \frac{\pi}{2} \pm \delta$, where $0 < \delta < \pi/2$ and $r_{0-} = \frac{\pi}{2} - \delta$ is the smallest positive root of $\sin r = \mu$. The r_{k-} cycles are stable (attracting from inside and outside), while the r_{k+} cycles are unstable (repelling from both inside and outside).
- $\mu = 0$: the origin is attracting, and for $k = 1, 2, \ldots$ there is a cycle $r = k\pi$ that is attracting if k is even and repelling if k is odd.
- $-1 < \mu < 0$: for each $k = 0, 1, 2, \ldots$, there are two radii $r_{k\pm} = 2k\pi + \frac{3\pi}{2} \pm \delta$, where $0 < \delta < \pi/2$ and $r_{0-} = \frac{3\pi}{2} - \delta$ is the smallest positive root of $\sin r = \mu$. The r_{k-} cycles are unstable (repelling from both inside and outside), while the r_{k+} cycles are stable (attracting from inside and outside).
- $\mu = -1$: then $\dot{r} = 0 \iff r = \frac{3\pi}{2} + 2k\pi$ for $k = 0, 1, 2, \ldots$, and all these radii are limit cycles attracting trajectories just outside themselves but repelling trajectories just inside.
- $\mu < -1$: then $\mu - \sin r < 0$, all r, so the origin $r = 0$ is attracting and there are no cycles.

These relationships are depicted in the seven plots in Figure 4 below.

From all this we conclude that global bifurcations of cycles occur at critical values $\mu = 1$, $\mu = 0$, and $\mu = -1$, for those are the values where cycles appear, disappear, or change stability. □

10. (Ex.8.4.3, p.322) Sketch the phase portraits of the system $\dot{x} = \mu x + y - x^2$, $\dot{y} = -x + \mu y + 2x^2$, which has a homoclinic bifurcation at $\mu_c \approx 0.066$, for values of μ just above and below μ_c.

    Solution: See the three phase portraits in Figure 2 corresponding to $\mu = -1 < \mu_c$, $\mu \approx \mu_c = 0.066$, and $\mu = 2 > \mu_c$. □



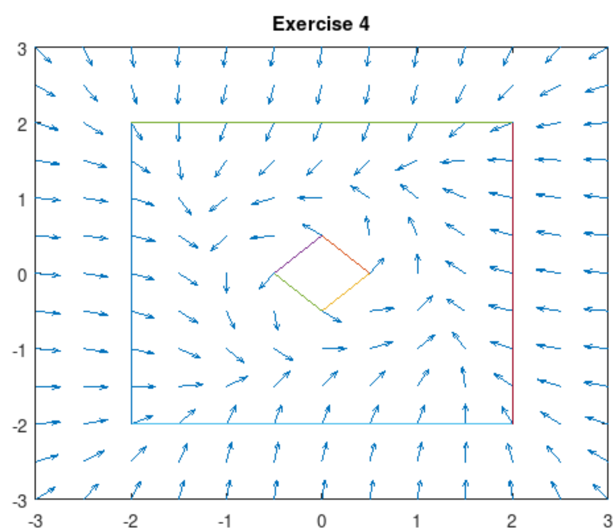


Figure 1: Flows and level curves for Exercise 2(c). Flows and trapping region for Exercise 4.

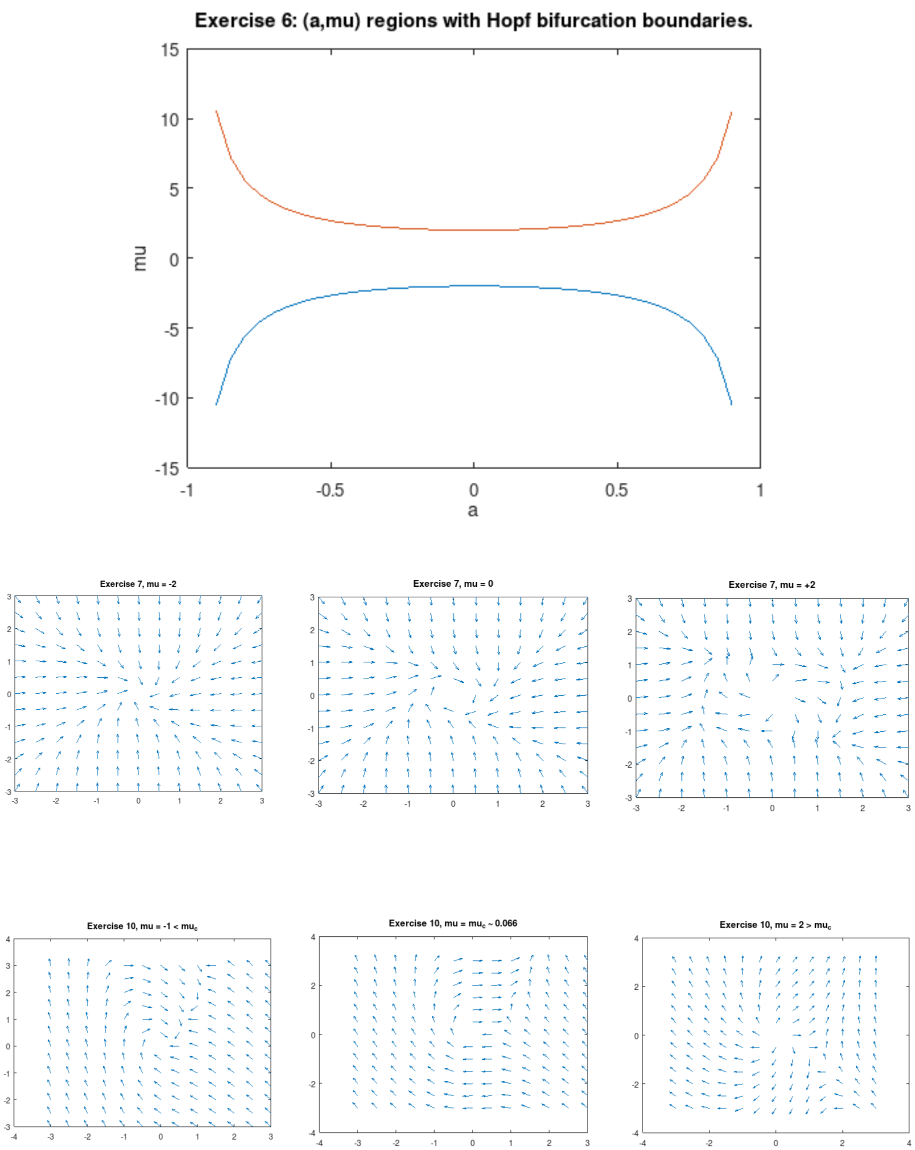


Figure 2: Hopf bifurcation regions and boundaries from Exercise 6. Flows for various μ near $\mu_c = 0$ from Exercise 7. Flows for various μ near $\mu_c = 0.066$ from Exercise 10.

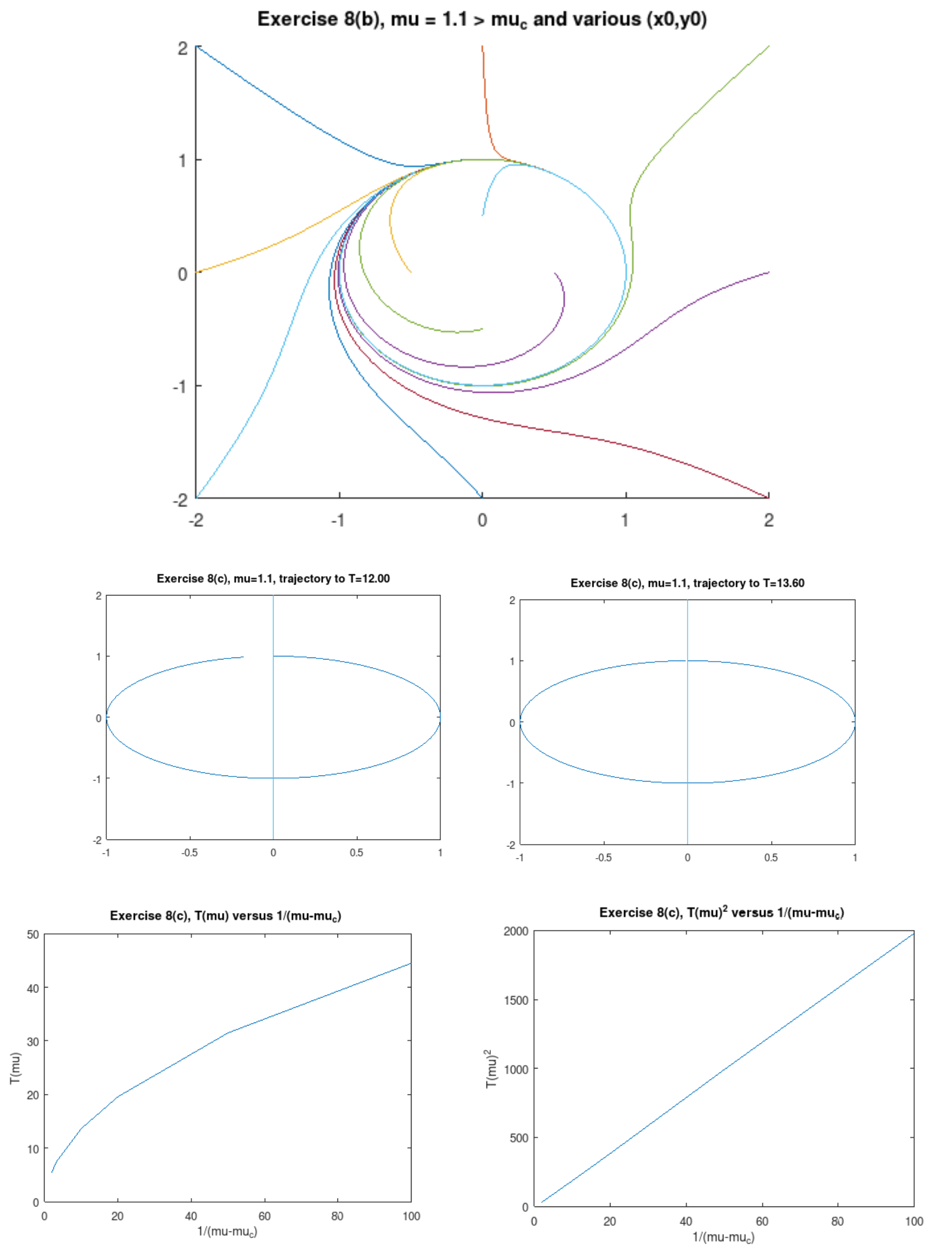


Figure 3: Trajectories from Exercise 8(b) for $\mu = 1.1$. Seeking the period with $T = 12.0$ and $T = 13.6$ in Exercise 8(c) for $\mu = 1.1$. Estimated period $T(\mu)$ versus $1/(\mu - mu_c)$ from Exercise 8(c).

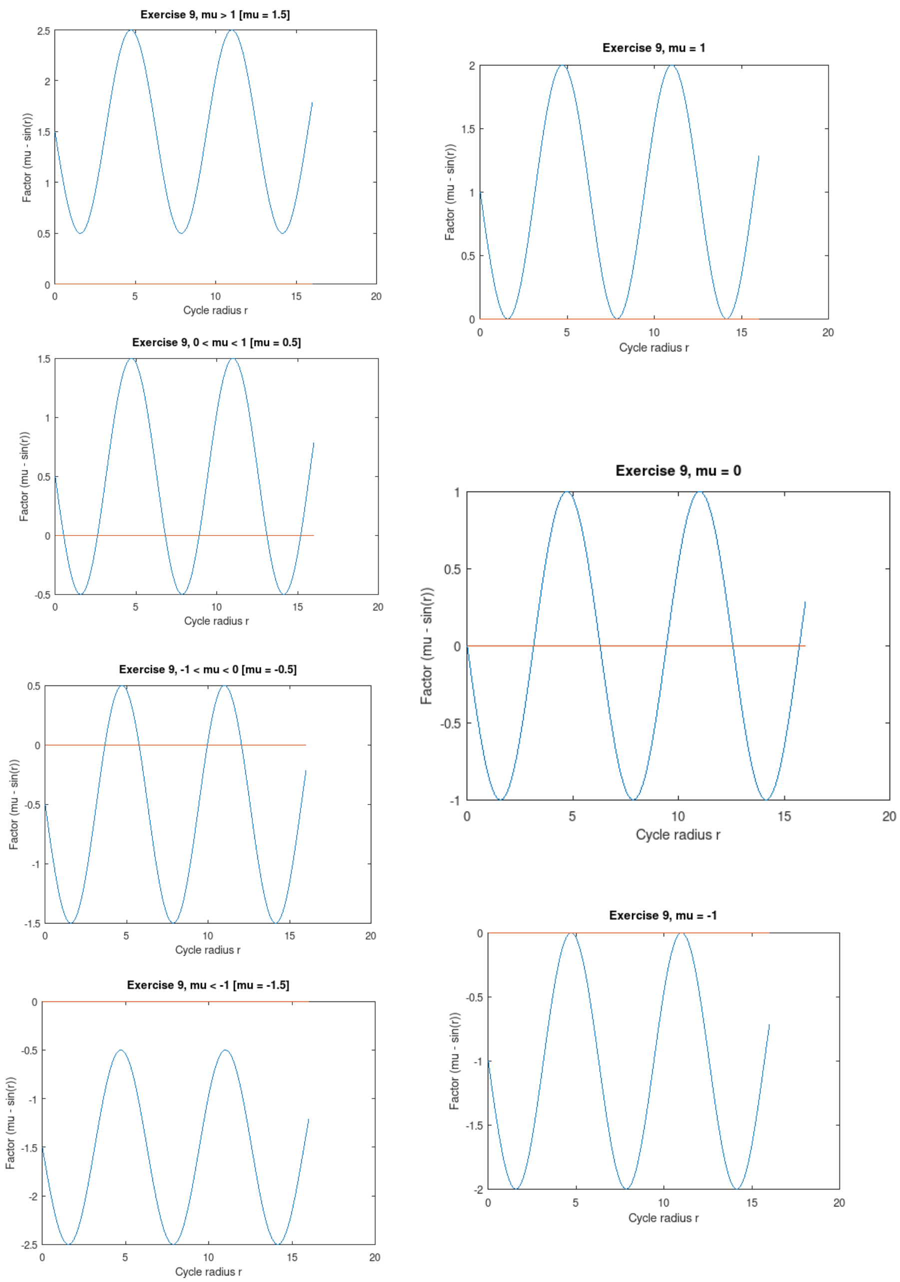


Figure 4: Factors from Exercise 9 for various values of μ in the range $-2 < \mu < 2$.