Ma 3520: Differential Equations and Dynamical Systems
Solutions to Homework Assignment 5

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Read Chapters 9 and 10 of the textbook, "Nonlinear Dynamics and Chaos," 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

1. (Ex.9.1.4,p.377) The Maxwell-Bloch equations for a laser are

$$\begin{aligned} \dot{E} &= \kappa(P - E) \\ \dot{P} &= \gamma_1(ED - P) \\ \dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP) \end{aligned}$$

[NOTE: there is a typo in the textbook at the P equation.]

(a) Find a fixed point with $E^* = 0$, show that it loses stability above some critical value $\lambda = \lambda_c$, and find λ_c.

(b) Classify the bifurcation at this λ_c.

(c) Find a change of variables that maps the system into the Lorenz system.

Solution: (a) If $E^* = 0$ at a fixed point (E^*, P^*, D^*), then the equations $(\dot{E}, \dot{P}, \dot{D}) = (0, 0, 0)$ force

$$E^* = 0, \quad P^* = 0, \quad D^* = \lambda + 1.$$

Find the bifurcation critical value $\lambda = \lambda_c$ by computing the Jacobian matrix at the fixed point:

$$J(E^*, P^*, D^*) = \begin{pmatrix} -\kappa & \kappa & 0 \\ \gamma_1 D & -\gamma_1 & \gamma_1 E \\ -\lambda\gamma_2 P & -\lambda\gamma_2 E & -\gamma_2 \end{pmatrix}; \qquad J_* = J(0, 0, \lambda + 1) = \begin{pmatrix} -\kappa & \kappa & 0 \\ \gamma_1(\lambda + 1) & -\gamma_1 & 0 \\ 0 & 0 & -\gamma_2 \end{pmatrix}$$

Linear stability is determined by the three eigenvalues of J_*. One of these is $-\gamma_2 < 0$. The other two are

$$\frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta}) = \frac{1}{2}\left(-\kappa - \gamma_1 \pm \sqrt{\kappa^2 + 2(2\lambda + 1)\gamma_1\kappa + \gamma_1^2}\right),$$

computed from 2×2 determininant $\Delta \stackrel{\text{def}}{=} -\gamma_1\kappa\lambda$ and trace $\tau \stackrel{\text{def}}{=} -\kappa - \gamma_1$.

Now Δ may be any sign since λ may be negative, positive, or zero. However $\tau < 0$, so there are three intervals for λ corresponding to different results for the remaining two eigenvalues:

- $\tau^2 - 4\Delta < 0$: two nonreal, complex conjugate eigenvalues giving **cycles**. Solve to get
$$\lambda < -\frac{1}{2} - \frac{\kappa^2 + \gamma_1^2}{4\kappa\gamma_1} < 0.$$

- $0 < \tau^2 - 4\Delta < \tau^2$: two real, negative eigenvalues giving a **(stable) attractor**. Solve to get
$$-\frac{1}{2} - \frac{\kappa^2 + \gamma_1^2}{4\kappa\gamma_1} < \lambda < 0.$$

- $\tau^2 - 4\Delta > \tau^2$: two real eigenvalues, one negative and one positive, giving an **(unstable) saddle**. Solve to get
$$\lambda > 0.$$

The critical value $\lambda = \lambda_c$ is found from the equation
$$\tau^2 - 4\Delta = \kappa^2 + 2(2\lambda + 1)\gamma_1\kappa + \gamma_1^2 = 0, \quad \Rightarrow \lambda_c = -\frac{1}{2} - \frac{\kappa^2 + \gamma_1^2}{4\gamma_1\kappa}.$$

(b) There is a **subcritical Hopf bifurcation** at $\lambda = \lambda_c < 0$ where the fixed point changes from stable attractor to a cycle. There is a **transcritical bifurcation** at $\lambda = 0$ where the fixed point changes from stable attractor to unstable saddle.

(c) There are evident similarities with the Lorenz equations so make the obvious substitutions
$$x(t) = E(t), \qquad y(t) = P(t/\gamma_1), \qquad z(t) = r - D(bt/\gamma_2),$$
then plug into the Lorenz system and derive the parameters (σ, b, r) from $(\kappa, \gamma_1, \gamma_2, \lambda)$ as follows:
$$\begin{aligned}
\dot{x} = \sigma(y - x) &\iff \dot{E} = \sigma(P - E) = \kappa(P - E), \qquad \Rightarrow \boxed{\sigma = \kappa}; \\
\dot{y} = (r - z)x - y &\iff \frac{1}{\gamma_1}\dot{P} = DE - P \iff \dot{P} = \gamma_1(DE - P); \\
\dot{z} = xy - bz &\iff -\frac{b}{\gamma_2}\dot{D} = EP - b(r - D) \iff \dot{D} = \gamma_2\left(r - D - \frac{1}{b}EP\right), \\
&\Rightarrow \boxed{b = 1/\lambda}, \quad \boxed{r = \lambda + 1}.
\end{aligned}$$

Of course, other substitutions will also work giving different parameter values. □

2. (Ex.9.2.3,p.378) Show that all trajectories of the Lorenz system eventually enter and remain inside a large sphere S of the form
$$V \stackrel{\text{def}}{=} x^2 + y^2 + (z - r - \sigma)^2 \; = \; C,$$
for sufficiently large $C > 0$. [HINT: show that V decreases along Lorenz trajectories for all (x, y, z) outside some fixed ellipsoid, then choose C large enough so that S encloses the ellipsoid.]

Solution: Start by computing $\frac{1}{2}\dot{V}$, scaled for convenience, along trajectories:

$$\begin{aligned}
\frac{1}{2}\dot{V} &= x\dot{x} + y\dot{y} + (z - r - \sigma)\dot{z} \\
&= x[\sigma(y - x)] + y[(r - z)x - y] + (z - r - \sigma)[xy - bz] \\
&= \sigma xy - \sigma x^2 + ry - xyz - y^2 + xyz - rxy - \sigma xy - bz^2 + rbz + \sigma bz \\
&= -\sigma x^2 - y^2 - bz^2 - rxy + ry + b[r + \sigma]z,
\end{aligned}$$

which is a quadratic function of x, y, z. Then $\dot{V} < 0$ whenever

$$\sigma x^2 + y^2 + bz^2 + rxy - ry - b[r + \sigma]z > 0,$$

which, after completing the squares and rotating the coordinates to eliminate cross terms, is the equation of an ellipsoid since the coefficients before x^2, y^2, z^2 are all the same sign, all positive.

Thus, outside the sphere $V = C^2$ of radius C sufficiently large to enclose that ellipsoid, the function V is strictly decreasing along trajectories. Conclude that all trajectories eventually enter the sphere. □

3. (Ex.9.3.1,p.379) Consider the system $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$ with constants ω_1, ω_2, for $(\theta_1(t), \theta_2(t))$ on the torus $[0, 1) \times [0, 1)$, namely taking values "modulo 1."

   This system is quasiperiodic but not periodic if ω_1/ω_2 is an irrational number. Find the largest Liapunov exponent of this system in that case.

   Solution: This system is described in section 8.6 on p.301 of our textbook. Trajectories are lines of slope $d\theta_2/d\theta_1 = \omega_2/\omega_1$ in the unit square, which represents the torus if its edges identified by left=right and top=bottom.

   To compute the Liapunov exponent, note that trajectories from any two initial points are parallel lines traversed at the same speed, so solutions neither separate nor approach each other. Conclude that the Liapunov exponent of this system is zero.

   NOTE: If the ratio ω_2/ω_1 is irrational, each of these wrapped lines becomes a dense set as $t \to \infty$. If the ratio is rational, then trajectories are closed cycles rather than dense sets. But in both cases, the distance between two solutions remains constant, equal to the distance between their initial points, so the Liapunov exponent is zero. □

4. (Ex.9.4.1,p.380) Using the codes in `lorenz.txt` on the class website, compute the Lorenz map for $r = 28$, $\sigma = 10$, and $b = 8/3$.

   Solution: See the top plot in Figure 1 below, produced by running the Octave code in `lorenz.txt` on the class website.

   There are 136 local maxima in the z coordinate of the solution for $0 \leq t \leq 100$, whose graph is displayed in the bottom plot of Figure 1 below. □

5. (Ex.9.5.[1,2,3],p.381) For $\sigma = 10$, $b = 8/3$, and each of the values of r given below, plot three graphs: $x(t)$, $y(t)$, and $(x(t), z(t))$, for a good range of times $t \geq 0$.

   (a) $r = 166.3$ (intermittent chaos),

   (b) $r = 212$ (noisy chaos),

   (c) selected values $145 < r < 166$ (period doubling).

   [HINT: use the codes in `lorenz.txt` or write your own.]

   Solution: See the plots in Figure 2 below. All flows were computed for $0 \leq t \leq 50$ with time step $h = 0.01$ using RK4. All systems used parameters $\sigma = 10$ and $b = 8/3$ but with varying Rayleigh numbers r.

   For (a), use $r = 166.3$. For (b), use $r = 212$.

   For (c), use the three values $r = 146$, $r = 151$, and $r = 165$, to explore the interval $145 < r < 166$. Any other values in this range are acceptable. □

6. (Ex.10.1.10,p.422) (a) Show that the map $x_{n+1} = 1 + \frac{1}{2}\sin x_n$ has a unique fixed point.

   (b) Determine whether the fixed point is stable.

   Solution: (a) Use the contraction mapping theorem: the iteration function $f(x) = 1 + \frac{1}{2}\sin x$ is continuously differentiable with derivative $f'(x) = \frac{1}{2}\cos x$ satisfying

$$|f'(x)| \leq \frac{1}{2} < 1$$

   for all $x \in \mathbf{R}$. Therefore there exists a unique fixed point.

   (b) The fixed point of a contraction map is stable as it is a global attractor. □

7. (Ex.10.1.11,p.422) Consider the map $x_{n+1} = 3x_n - x_n^3$.

   (a) Find all fixed points and classify them by stability.

   (b) Draw a cobweb starting at $x_0 = 1.9$. Then prove that x_n remains bounded for all $n = 1, 2, \ldots$.

   (c) Draw a cobweb starting at $x_0 = 2.1$. Then prove that $|x_n| \to \infty$ as $n \to \infty$.

Solution: (a) Solve for x_* in

$$x_* = f(x_*) \stackrel{\text{def}}{=} 3x_* - x_*^3, \qquad \iff 0 = 2x_* - x_*^3 = (2 - x_*^2)x_*$$

to get $x_* \in \{0, \sqrt{2}, -\sqrt{2}\}$.

Check for linear stability at each fixed point using $f'(x) = 3 - 3x^2 = 3(1 - x^2)$:

- $f'(0) = 3 > 0$, **unstable**,
- $f'(\sqrt{2}) = -3 < 0$, **stable**,
- $f'(-\sqrt{2}) = -3 < 0$, **stable**.

(b) See the cobweb plot in Figure 3 below. It suggests that iterations from $x_0 = 1.9$ remain within the interval $[-2, 2]$.

To prove this conjecture, note that $-2 \le x \le 2 \Rightarrow -2 \le f(x) \le 2$, so the iteration from any $x_0 \in [-2, 2]$ remains confined to that interval.

(c) See the cobweb plot in Figure 3 below. It suggests that iterations from $x_0 = 2.1$ diverge with $|x_n| \to \infty$ as $n \to \infty$ but with alternating signs.

To prove this conjecture, first note that $x < -2 \Rightarrow f(x) > 2$, while $x > 2 \Rightarrow f(x) < -2$. To show divergence, consider the ratio

$$\left| \frac{f(x_n)}{x_n} \right| = \left| \frac{3x_n - x_n^3}{x_n} \right| = \left| 3 - x_n^2 \right| > 1,$$

if $|x_n| > 2$. If $|x_0| = 2.1$, then the ratio is 1.41 so $|x_n|$ increases exponentially as $n \to \infty$ with an increasing rate that is at least 1.41.

NOTE: Due to this rapid divergence, only a few steps in the cobweb should be plotted. □

8. (Ex.10.1.12,p.423) Newton's method finds the root of an equation $g(x) = 0$ for a differentable function g by iteration of the *Newton map*

$$f(x) = x - \frac{g(x)}{g'(x)}.$$

(a) Apply the method to the function $g(x) = x^2 - 4$.

(b) Show that the Newton map in part (a) has stable fixed points $x^* = \pm 2$.

Solution: (a) Straightforward:

$$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2 - 4}{2x} = \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x}.$$

(b) Check for stability at a fixed point x_* by evaluating $|f'(x)|$. Note that

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} < 1/2, \qquad f'(x) > -1 \iff x^2 > \frac{4}{3},$$

and $f(x)$ has the same sign as x, so f preserves and is a contraction map on each of the two intervals $(-\infty, -2/\sqrt{3})$ and $(2/\sqrt{3}, \infty)$. Since $2/\sqrt{3} \approx 1.1547$, each fixed point is in one of these intervals, so both fixed points are stable. □

9. (Ex.10.2.2,p.423) Use a cobweb to show that $x_* = 0$ is a globally stable fixed point for the logistic map with $0 \leq r \leq 1$.

Solution: Let $f(x) = rx(1-x)$ be the logistic map to iterate. Then $x_* = 0$ is a fixed point since $f(0) = 0$ for any r.

If $r = 0$ then $f(x) = 0$ for any x, so $x_* = 0$ is clearly a stable fixed point. Thus we may restrict our attention to $r > 0$.

Any other fixed point x_* satisfies

$$f(x_*) = x \Rightarrow x_* = rx_*(1-x_*) \Rightarrow x_* = \frac{r-1}{r} \leq 0,$$

with equality iff $r = 1$. Hence $x_* = 0$ is the unique fixed point for the logistic map iteration with $0 \leq r \leq 1$.

To show that it is a stable fixed point, first check if f is a contraction map. Compute

$$f'(x) = r(1-2x), \quad \Rightarrow |f'(x)| \leq r,\ 0 \leq x \leq 1.$$

Thus for $0 \leq r < 1$, f is a contraction on the whole interval $x \in [0,1]$, so the contraction mapping theorem implies that its unique fixed point $x_* = 0$ is stable.

For the endpoint $r = 1$, note that $0 \leq x(1-x) \leq 1/4$ for any $0 \leq x \leq 1$, so $x_n \in [0, 1/4]$ for $n > 1$ with any $x_0 \in [0,1]$. Also, if $0 < x_n \leq 1/4$,

$$0 \leq x_{n+1} = x_n(1-x_n) = x_n - x_n^2 < x_n$$

so the iteration is strictly decreasing on $0 < x < 1$. Thus for any $x_0 \in [0,1]$, the greatest lower bound (or *infimum*) of $\{x_n : n = 1, 2, \ldots\}$ cannot be positive or negative, and therefore must be 0. Conclude that 0 is a stable fixed point.

Cobwebs from several initial values $x_0 \in [0,1]$ with $r = 1$ are depicted in Figure 5. □

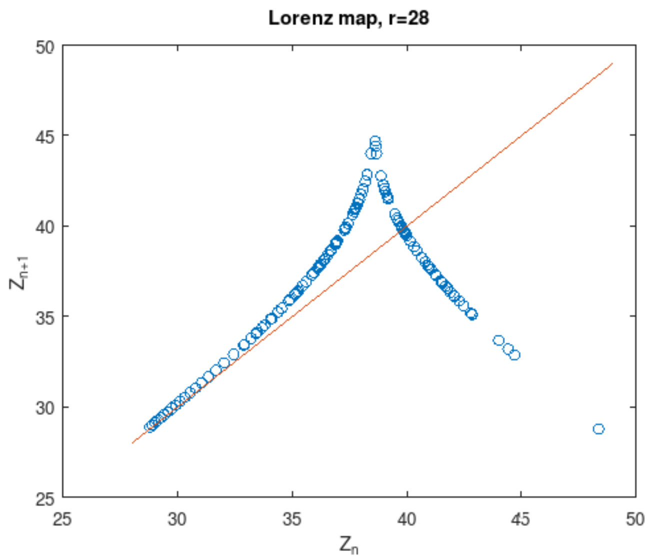
10. (Ex.10.2.3,p.423) Compute an orbit diagram for the logisitic map over the range $3.4 \leq r \leq 3.7$ using increments in r chosen to make it look nice.

Solution: See the graphs in Figure 5 produced by the code in `hw5-plot.txt` on the class website.

(1) and (2) are experiments to determine how many iterations per r value produce an acceptable depiction of the orbit. The code ran without noticeable delay.

(3) is the final result with dr=0.0001 spacing in r values and norbit=300 points per orbit. That computation took about 5 seconds to complete.

Any of these three plots are "nice" enough for full credit as they all exhibit period doubling bifurcations and chaotic behavior. $\square$



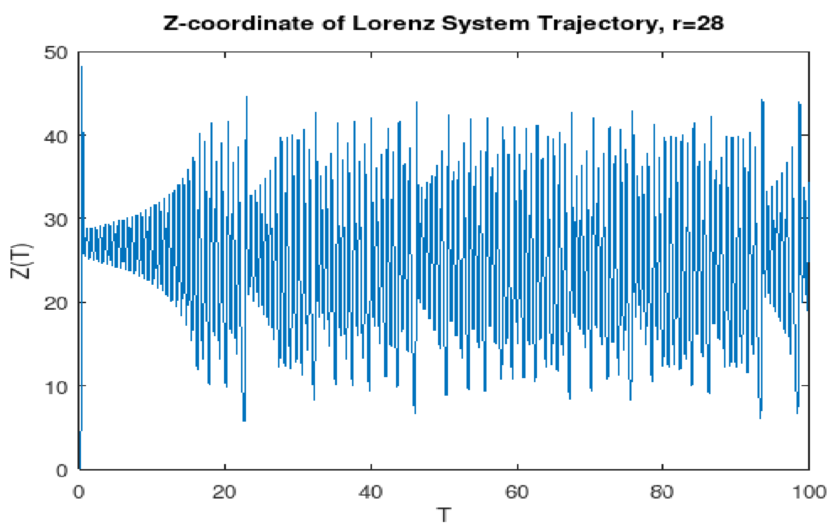


Figure 1: Lorenz map, and a plot of the Z-coordinate producing the Lorenz map for Exercise 4.

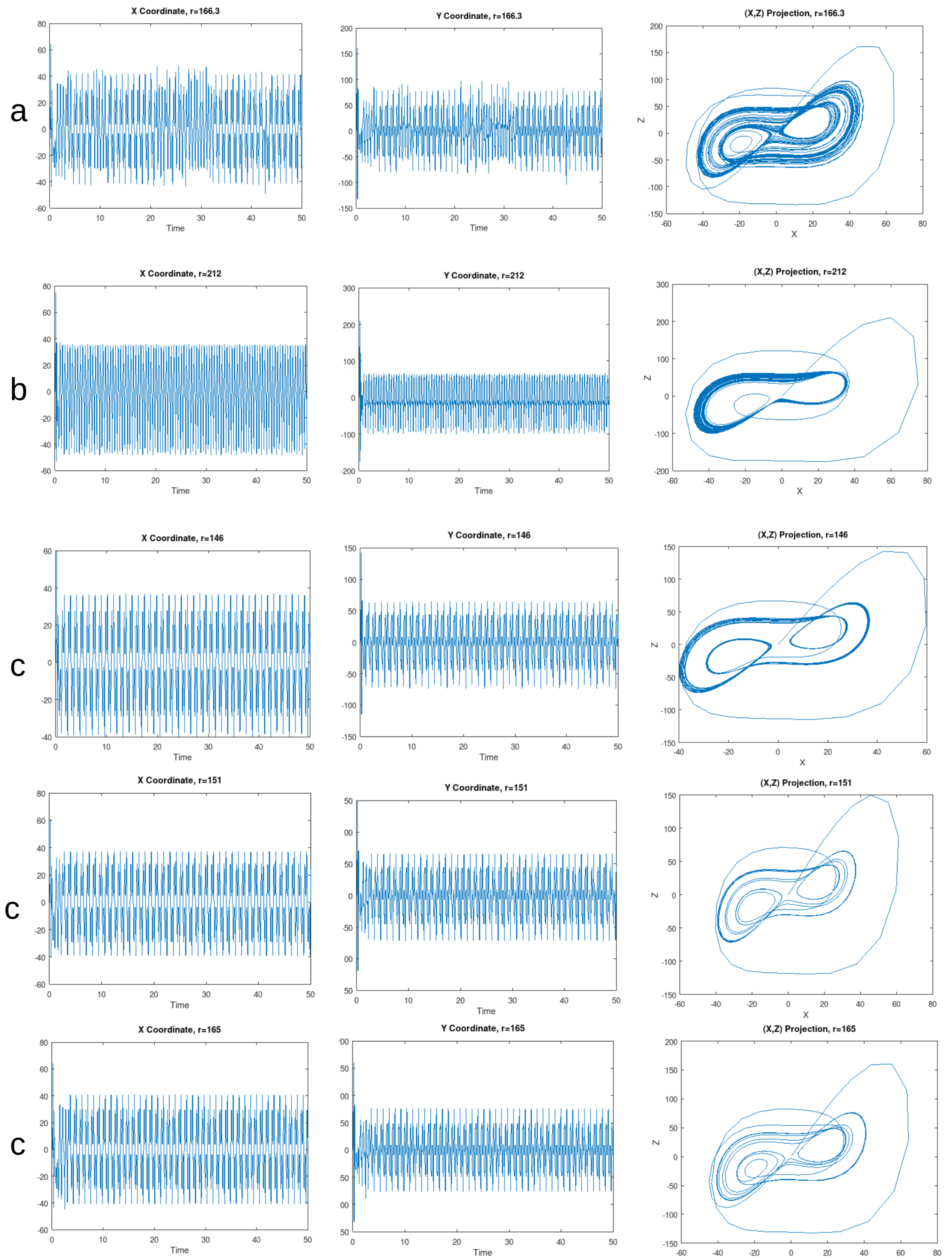
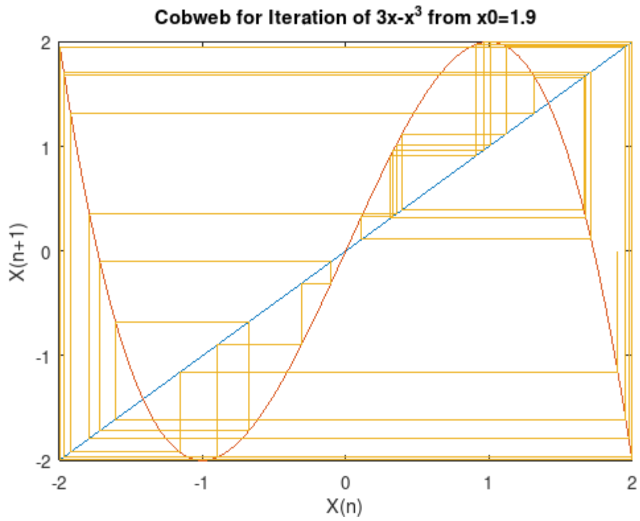
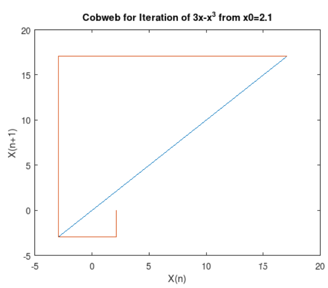


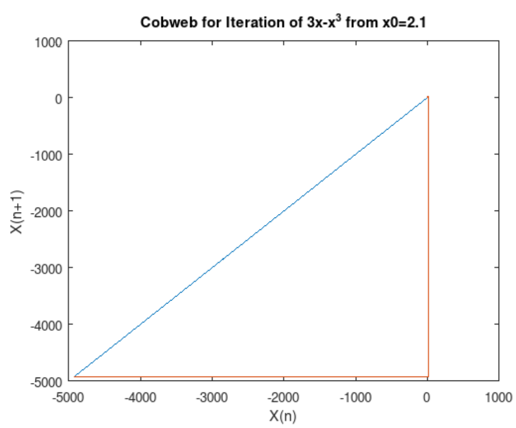
Figure 2: Chaos in the Lorenz system: X, Y, and (X,Z) plots for Rayleigh numbers (a) 166.3, (b) 212, and (c) (146, 151, 165) for Exercise 5(abc).

(b)





2 steps



(c)

3 steps

Figure 3: Cobweb plots for the iteration of $f(x) = 3x - x^3$ from initial points (b) $x_0 = 1.9$ (30 steps, bounded) and (c) $x_0 = 2.1$ (2 and 3 steps, diverging) for Exercise 7(bc).

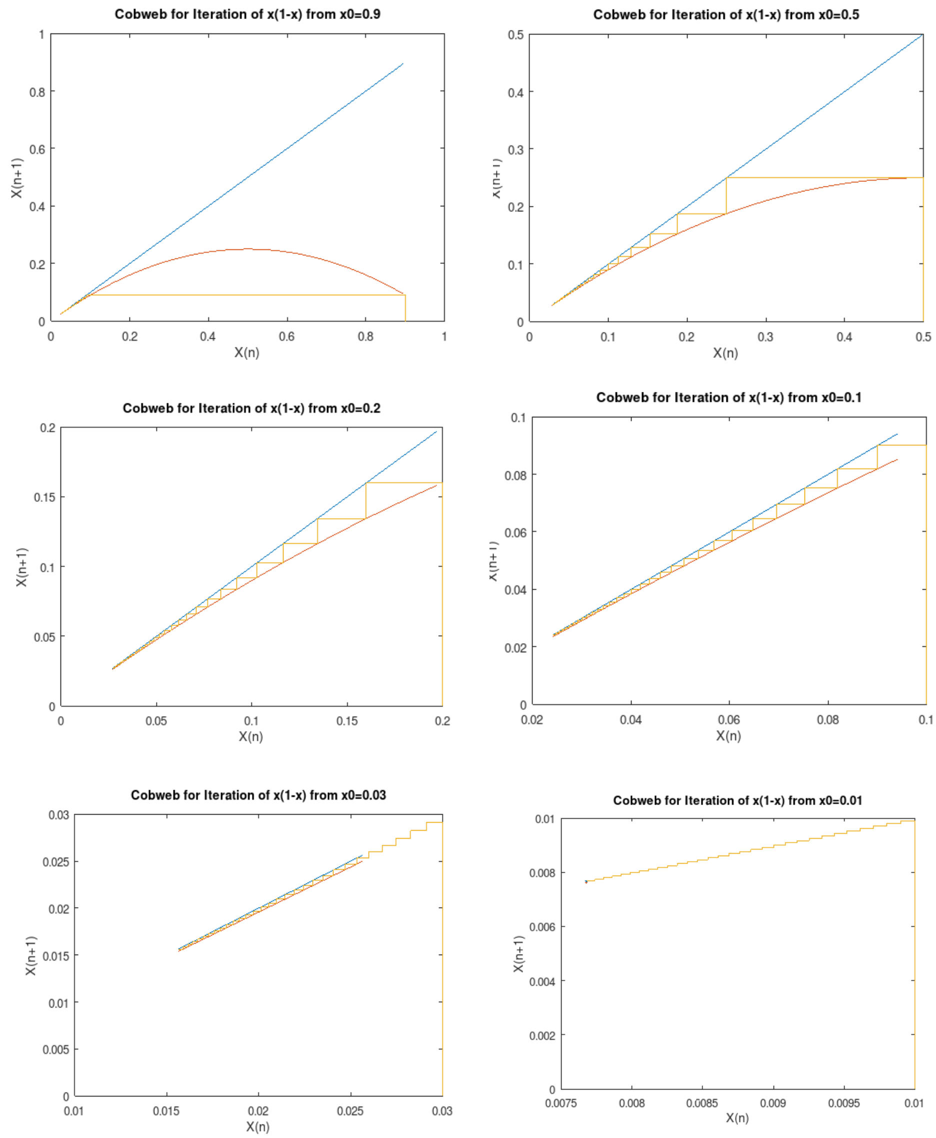


Figure 4: Exercise 9: Cobweb plots with 30 steps for the logistic map $f(x) = x(1-x)$ for various initial points $x_0 \in (0,1)$, showing slow convergence near the fixed point $x_* = 0$.

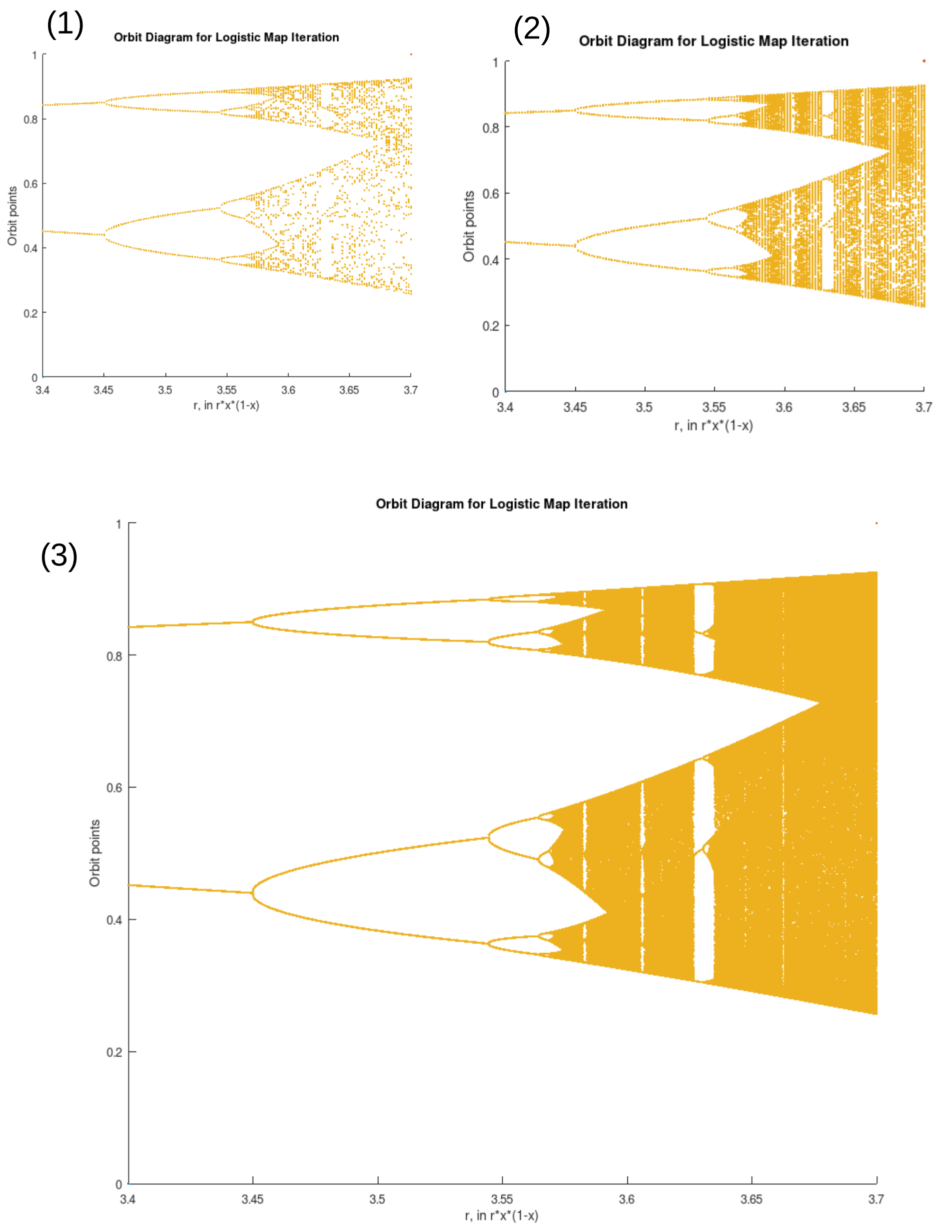


Figure 5: Exercise 10: Orbit diagrams for the logistic map $f(x) = rx(1-x)$ for values $3.4 \leq r \leq 3.7$, sampling orbits after ntrans=1000 initial iterations to escape transients. (1) dr=0.002, norbit=32. (2) dr=0.002, norbit=300. (3) dr=0.0001, norbit=300.