

Ma 3520: Differential Equations and Dynamical Systems

Solutions to Homework Assignment 6

Prof. Wickerhauser

Read Chapters 11 and 12 of the textbook, “Nonlinear Dynamics and Chaos,” 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

- (Ex.11.1.6,p.453) The *decimal shift map* is $x_{n+1} = 10x_n \pmod{1}$. Initial values x_0 are decimal expansions of real numbers in $[0, 1]$.
 - Show that the map has countably many periodic orbits.
 - Show that the map has uncountably many aperiodic orbits.
 - An *eventually fixed point* of an iteration is an initial value x_0 such that, for some $N \geq 0$, we have $x_n = x_N$ for all $n \geq N$. Is the number of eventually fixed points for the decimal shift map countable or uncountable?

Solution: First recall that $x \in [0, 1]$ has an eventually-repeating decimal expansion if and only if x is rational.

- There are countably many rational numbers in $[0, 1]$, and they and only they give periodic orbits.
 - There are uncountably many irrational numbers in $[0, 1]$, and they each give an aperiodic orbit, and distinct irrationals give distinct orbits, hence there are uncountably many aperiodic orbits.
 - There are only countably many eventually fixed points x_0 since they are a subset of the rationals in $[0, 1]$. \square
- (Ex.11.2.3,p.454) The *outer measure* of a subset $S \subset \mathbf{R}$ is defined by

$$\mu(S) \stackrel{\text{def}}{=} \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : S \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\},$$

namely the greatest lower bound of the sum of the lengths of a countable cover of S by open intervals $\{(a_i, b_i) : i = 1, 2, \dots\}$.

Show that any countable subset $S = \{x_1, x_2, \dots\}$ has outer measure zero.

Solution: Let $\epsilon > 0$ be given, and let $G_i \stackrel{\text{def}}{=} (x_i - \epsilon/2^i, x_i + \epsilon/2^i)$ be a small open interval containing x_i , for $i = 1, 2, \dots$. Notice that $\mu(G_i) = 2\epsilon/2^i$. Then $S \subset \bigcup_i G_i$, so

$$\mu(S) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon,$$

and since $\epsilon > 0$ was arbitrary, conclude that $\mu(S) = 0$. \square

3. (Ex.11.3.1,p.455) The *middle halves Cantor set* H is a fractal constructed in stages from the interval $S_0 = [0, 1]$ by removing the middle half of each subinterval S_n to get S_{n+1} , then letting

$$H \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} S_n$$

- (a) Find the similarity dimension of H .
 (b) Find the box dimension of H .
 (c) Find the measure of H .

Solution: (a) Scale factor $r = 4$, replication factor $m = 2$, so the similarity dimension is

$$d_{\text{SIM}} = \frac{\log m}{\log r} = \frac{\log 2}{\log 4} = \frac{\log 2}{2 \log 2} = \frac{1}{2}.$$

- (b) “Boxes” here are line segments, and at stage n it takes $N(\epsilon) = 2^n$ of them, each of length $\epsilon = 1/4^n$, to cover the 2^n intervals comprising S_n . Hence

$$d_{\text{BOX}} = \frac{\log N(\epsilon)}{\log(1/\epsilon)} = \frac{\log 2^n}{\log 4^n} = \frac{n \log 2}{2n \log 2} = \frac{1}{2}.$$

- (c) The measure of S_n is $1/2^n$, and $H \subset S_n$ for every $n = 0, 1, 2, \dots$, so the measure of H is zero. \square

4. (Ex.11.3.[8,9],p.456) Find the similarity dimension of:

- (a) the Sierpinski carpet in \mathbf{R}^2 .
 (b) the Menger sponge in \mathbf{R}^3 ,
 (c) the Menger sponge in \mathbf{R}^N , $N > 3$.

Solution: (a) Scale factor $r = 3$, replication factor $m = 8$, so the similarity dimension is

$$d_{\text{SIM}} = \frac{\log m}{\log r} = \frac{\log 8}{\log 3} \approx 1.8928.$$

- (b) Scale factor $r = 3$, replication factor $m = 20$, so the similarity dimension is

$$d_{\text{SIM}} = \frac{\log m}{\log r} = \frac{\log 20}{\log 3} \approx 2.7268.$$

- (c) Scale factor $r = 3$. In R^N , there are 3^N sub-hypercubes after rescaling, from which the central one and the $2N$ in the face centers are removed, leaving a replication factor $m = 3^N - 2N - 1$, so the similarity dimension is

$$d_{\text{SIM}} = \frac{\log(3^N - 2N - 1)}{\log 3},$$

which asymptotically increases to N as $N \rightarrow \infty$. \square

5. (Ex.11.3.[8,9],p.456) Find the box dimension of:

- (a) the Sierpinski carpet in \mathbf{R}^2 .
 (b) the Menger sponge in \mathbf{R}^3 ,

(c) the Menger sponge in \mathbf{R}^N , $N > 3$.

Solution: (a) Let S_n be the n th stage of construction, so $\bigcap_n S_n$ is the limit fractal. “Boxes” here are squares, and at stage n it takes $N(\epsilon) = 8^n$ of them, each of side $\epsilon = 1/3^n$, to cover the 8^n subsquares comprising S_n . Hence

$$d_{\text{BOX}} = \frac{\log N(\epsilon)}{\log(1/\epsilon)} = \frac{\log 8^n}{\log 3^n} = \frac{\log 8}{\log 3} \approx 1.8928.$$

NOTE: Since $d_{\text{BOX}} < 2$, the measure (or area) of this set is zero.

(b) Let M_n be the n th stage of construction, so $\bigcap_n M_n$ is the limit fractal. “Boxes” here are cubes, and at stage n it takes $N(\epsilon) = 20^n$ of them, each of side $\epsilon = 1/3^n$, to cover the 20^n subsquares comprising M_n . Hence

$$d_{\text{BOX}} = \frac{\log N(\epsilon)}{\log(1/\epsilon)} = \frac{\log 20^n}{\log 3^n} = \frac{\log 20}{\log 3} \approx 2.7268.$$

NOTE: Since $d_{\text{BOX}} < 3$, the measure (or volume) of this set is zero.

(c) Let M_n^N be the n th stage of construction, so $\bigcap_n M_n^N$ is the limit fractal. “Boxes” here are hyper-cubes, and at stage n it takes $N(\epsilon) = (3^N - 2N - 1)^n$ of them, each of side $\epsilon = 1/3^n$, to cover the subsquares comprising M_n^N . Hence

$$d_{\text{BOX}} = \frac{\log N(\epsilon)}{\log(1/\epsilon)} = \frac{\log(3^N - 2N - 1)^n}{\log 3^n} = \frac{\log(3^N - 2N - 1)}{\log 3}.$$

NOTE: Since $d_{\text{BOX}} < N$, the measure (or N -volume) of this set is zero. □

6. (Ex.12.1.8,p.488) Hénon’s area-preserving quadratic map is the iteration

$$\begin{aligned} x_{n+1} &= x_n \cos \alpha - (y_n - x_n^2) \sin \alpha \\ y_{n+1} &= x_n \sin \alpha + (y_n - x_n^2) \cos \alpha \end{aligned}$$

where α is a fixed parameter satisfying $0 \leq \alpha \leq \pi$.

(a) Show that the map is area-preserving.

(b) Find the inverse mapping.

(c) Plot at least 10,000 iterations from various (x_0, y_0) in the square $-1 \leq x, y \leq 1$ for various α including $\cos \alpha = 0.24$.

Solution: (a) Compute the Jacobian

$$J(x, y) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

and note that it has determinant $\cos^2 \alpha + \sin^2 \alpha = 1$ at every (x, y) , regardless of α .

(b) Note that

$$x_{n+1} \cos \alpha + y_{n+1} \sin \alpha = x_n(\cos^2 \alpha + \sin^2 \alpha) = x_n,$$

and likewise

$$y_{n+1} \cos \alpha - x_{n+1} \sin \alpha = y_n - x_n^2 = y_n - (x_{n+1} \cos \alpha + y_{n+1} \sin \alpha)^2$$

after back substitution. Thus

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \cos \alpha + y_{n+1} \sin \alpha \\ y_{n+1} \cos \alpha - x_{n+1} \sin \alpha + (x_{n+1} \cos \alpha + y_{n+1} \sin \alpha)^2 \end{pmatrix}$$

Conclude that Hénon's area-preserving map is invertible.

(c) See Figure 1 below for a number of plots with 20,000 points for three values of α and various initial points (x_0, y_0) . □

7. (Ex.12.1.9,p.488) The so-called *standard map* is the iteration

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1} \pmod{2\pi}, \\ y_{n+1} &= y_n + k \sin x_n \pmod{2\pi}, \end{aligned}$$

where $k \geq 0$ is a nonlinearity parameter. Notice that y_{n+1} must be computed first. It maps the square $S = [0, 2\pi] \times [0, 2\pi]$ into itself.

(a) Show that the map is area-preserving for all k .

(b) Plot at least 10,000 iterations from various (x_0, y_0) in the square S for each $k \in \{0, 0.5, 1, 2\}$.

Solution: Substitute for y_{n+1} in the first equation to have (x_{n+1}, y_{n+1}) solely in terms of (x_n, y_n) :

$$\begin{aligned} x_{n+1} &= x_n + y_n + k \sin x_n \pmod{2\pi}, \\ y_{n+1} &= y_n + k \sin x_n \pmod{2\pi}. \end{aligned}$$

(a) Compute the Jacobian matrix of the map underlying the iteration:

$$J = \begin{pmatrix} 1 + k \cos x & 1 \\ k \cos x & 1 \end{pmatrix}; \quad \det J = (1 + k \cos x)(1) - (k \cos x)(1) = 1.$$

Since the Jacobian determinant is 1, conclude that the iteration's map is area-preserving.

(b) For $k = 0$ the mapping simplifies to

$$\begin{aligned} x_{n+1} &= x_n + y_0 \pmod{2\pi}, \\ y_{n+1} &= y_0 \pmod{2\pi}. \end{aligned}$$

Its orbit from (x_0, y_0) consists of the points

$$\{(x_0 + ny_0 \pmod{2\pi}, y_0) : n = 0, 1, 2, \dots\}$$

which lie on a horizontal line segment in S . If y_0/π is rational, then there will be finitely many points. Otherwise, there will be infinitely many points and the orbit will be dense in the line segment $[0, 2\pi] \times \{y_0\}$

See Figure 2 below for a number of plots with 20,000 points for the remaining three values of k and various initial points (x_0, y_0) . □

8. (Ex.12.2.4,p.489) Find all fixed points for the Hénon map and show that they exist only if $a > a_0$ for some a_0 , and then determine a_0 .

Solution: Any fixed point for the Hénon map, with fixed parameters a, b , must satisfy:

$$\begin{aligned}x_n &= x_{n+1} = y_n + 1 - ax_n^2, \\y_n &= y_{n+1} = bx_n,\end{aligned}$$

from which back substitution and writing $x = x_n = x_{n+1}$ yields

$$x = bx + 1 - ax^2 \iff ax^2 + (1-b)x - 1 = 0 \iff x = \frac{b-1 \pm \sqrt{(b-1)^2 + 4a}}{2a}.$$

This will have a real solution only if $(b-1)^2 + 4a \geq 0$, or equivalently only if

$$a \geq a_0 \stackrel{\text{def}}{=} -\frac{(b-1)^2}{4}.$$

End case $a = a_0$ yields a unique fixed point:

$$x_* = \frac{b-1}{2a} = \frac{2(b-1)}{(b-1)^2} = \frac{2}{b-1}; \quad y_* = bx_* = \frac{2b}{b-1}.$$

Other cases $a > a_0$ yield this pair of fixed points:

$$x_*^\pm = \frac{b-1 \pm \sqrt{(b-1)^2 + 4a}}{2a}; \quad y_*^\pm = bx_*^\pm = \frac{b^2 - b \pm \sqrt{b^2(b-1)^2 + 4ab^2}}{2a}.$$

□

9. (Ex.12.2.9,p.490) Let T be the Hénon map with parameters $a = 1.4$ and $b = 0.3$. Let $ABCD$ denote the quadrilateral with vertices $A = (-1.33, 0.42)$, $B = (1.32, 0.133)$, $C = (1.245, -0.14)$, and $D = (-1.06, -0.5)$.

(a) Plot $ABCD$ and its image $T(ABCD)$. [HINT: represent the edges of $ABCD$ parametrically and note that T maps line segments to parabolic arcs.]

(b) Prove that $T(ABCD) \subset ABCD$. Conclude that $ABCD$ is a trapping region for T .

Solution: (a) See Figure 2 below.

(b) Recall that a point $Y \in \mathbf{R}^2$ is *inside* the half-plane determined by a parameterized line $PQ = \{P + \mathbf{v}t : t \in \mathbf{R}\}$ if and only if the inner product $(Y - P) \cdot \mathbf{n} < 0$, where $\mathbf{n} = (n_x, n_y)$ is an outward-pointing normal vector perpendicular to the direction vector $\mathbf{v} = (v_x, v_y)$. But that normal has coordinates $(n_x, n_y) = \pm(v_y, -v_x)$, with the sign being chosen so that $P + \mathbf{n}$ is outside the half-plane.

Quadrilateral $ABCD$ is the intersection of four half-planes:

- Below AB : direction vector $\mathbf{v}_{AB} = (2.650, -0.287)$, outward normal $\mathbf{n}_{AB} = (0.287, 2.650)$, parametrized boundary line segment $AB = \{A + \mathbf{v}_{AB}t : 0 \leq t \leq 1\}$;
- Left of BC : direction vector $\mathbf{v}_{BC} = (-0.075, -0.273)$, outward normal $\mathbf{n}_{BC} = (0.273, -0.075)$, parametrized boundary line segment $BC = \{B + \mathbf{v}_{BC}t : 0 \leq t \leq 1\}$;
- Above CD : direction vector $\mathbf{v}_{CD} = (-2.305, -0.360)$, outward normal $\mathbf{n}_{CD} = (0.360, -2.305)$, parametrized boundary line segment $CD = \{C + \mathbf{v}_{CD}t : 0 \leq t \leq 1\}$;
- Right of BC : direction vector $\mathbf{v}_{DA} = (-0.270, 0.920)$, outward normal $\mathbf{n}_{DA} = (-0.920, -0.270)$, parametrized boundary line segment $DA = \{D + \mathbf{v}_{DA}t : 0 \leq t \leq 1\}$.

It suffices to show that if $Y \in AB \cup BC \cup CD \cup DA$, then $T(Y)$ is inside all four half-planes above, which requires 16 inner products and sign tests in total. Because T is continuous, and thus maps interiors to interiors, we may conclude that $T(ABCD) \subset ABCD$ for all interior points as well. \square

10. (Ex.12.3.2,p.491)

- (a) Find the fixed points of the Rössler system, stating conditions on a, b, c under which they exist.
- (b) Fix $a = b = 0.2$ and plot a partial bifurcation diagram in terms of parameter c .
- (c) Find a trapping region for the system at your choice of a, b, c .

Solution: Recall that the Rössler system is

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}$$

where a, b, c are fixed constants.

(a) Solve $(\dot{x}, \dot{y}, \dot{z}) = (0, 0, 0)$ to find fixed points:

$$\begin{aligned}0 &= -y - z \iff z = -y; \\ 0 &= x + ay \iff x = -ay \\ 0 &= b + z(x - c) \iff x = -\frac{b}{z} + c = \frac{b}{y} + c\end{aligned}$$

Substituting the second equation into the third gives

$$-ay = \frac{b}{y} + c \iff ay^2 + cy + b = 0.$$

There are several cases to consider:

1. $a = b = c = 0$: then any $y \in \mathbf{R}$ is a solution, giving the non-isolated set of fixed points $(0, y, -y)$.
2. $a = 0, c \neq 0, \text{ any } b \in \mathbf{R}$: then $y = -b/c$ gives the unique fixed point $(0, -b/c, b/c)$.
3. $a \neq 0$: The quadratic equation for y has real roots iff $c^2 - 4ab \geq 0$, otherwise there are no fixed points. There are two further cases to consider:
 - 3a. $c^2 - 4ab = 0$: unique repeated root $y = -c/(2a)$ gives the unique fixed point $(\frac{c}{2}, -\frac{c}{2a}, \frac{c}{2a})$.
 - 3b. $c^2 - 4ab > 0$: distinct roots $y_{\pm} = [-c \pm \sqrt{c^2 - 4ab}]/(2a)$ give two fixed points $(-ay_+, y_+, -y_+)$ and $(-ay_-, y_-, -y_-)$. For fixed a, b , these two points move farther apart as $|c|$ increases.

To classify these fixed points, compute the Jacobian matrix J , its determinant Δ , and its trace τ :

$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{pmatrix}; \quad \Delta = az + x - c, \quad \tau = x + a - c.$$

For the cases listed above, linearization classifies the isolated fixed points as follows:

2. $a = 0, c \neq 0, \text{ any } b \in \mathbf{R}$: $\Delta = -c, \tau = -c$, **saddle** if $c > 0$, or **repeller** if $c < 0$.
3. $a \neq 0, c^2 \geq 4ab$:

- 3a. $c^2 = 4ab$: $\Delta = 0$, $\tau = a - c/2$, **degenerate attractor** if $a < c/2$, $b > c/2$, or **degenerate repeller** if $a > c/2$, $b < c/2$.
- 3b. $c^2 > 4ab$: either $\Delta_+ = -\sqrt{c^2 - 4ab} < 0$ and $\tau_+ = \frac{1}{2}(2a - c - \sqrt{c^2 - 4ab})$, which is a **saddle**, or else $\Delta_- = \sqrt{c^2 - 4ab} > 0$ and $\tau_- = \frac{1}{2}(2a - c + \sqrt{c^2 - 4ab})$, which is an **attractor or a repeller** depending upon the sign of τ_- .

(b) For $a = b = 0.2$, we are in Case 3 from part (a). There is a **saddle node bifurcation** where $c^2 - 4ab = 0$, namely at the critical value $c_B \stackrel{\text{def}}{=} 2\sqrt{ab} = 0.4$ where $X_B^* = (0.2, -1, 1)$.

The two fixed points that appear as c increases across c_B are

$$X_{\pm}^* = (-ay_{\pm}, y_{\pm}, -y_{\pm}) = (0.2, -1, 1) \left[\frac{c \mp \sqrt{c^2 - 0.16}}{0.4} \right]$$

As seen in part (a), X_+^* is a **saddle** while X_-^* will be a **repeller** for nearby $c > c_B = 0.4$ because for these values

$$\tau_- = \frac{1}{2} \left(0.4 - c + \sqrt{c^2 - 0.16} \right) = \frac{1}{2} \sqrt{c - 0.4} \left[\sqrt{c + 0.4} - \sqrt{c - 0.4} \right] > 0$$

A plot of any single coordinate of X_B^* and X_{\pm}^* versus c near c_B would suffice as a partial bifurcation diagram.

Plotting orbits (or just one coordinate) for large values of c will show the chaotic behavior.

(c) One good choice is $a = b = 0.2$ and $c = 5.7$ as in the textbook's example of chaos. Examining a few flows from $X_0 = (1, 1, 1)$ it seems that $-15 < x < 15$, $-10 < y < 10$, and $0 < z < 30$ is a trapping region. For this exercise, it suffices to compute some long-term trajectories from a sampling of points inside this region and confirm that the flows remain within these bounds. \square

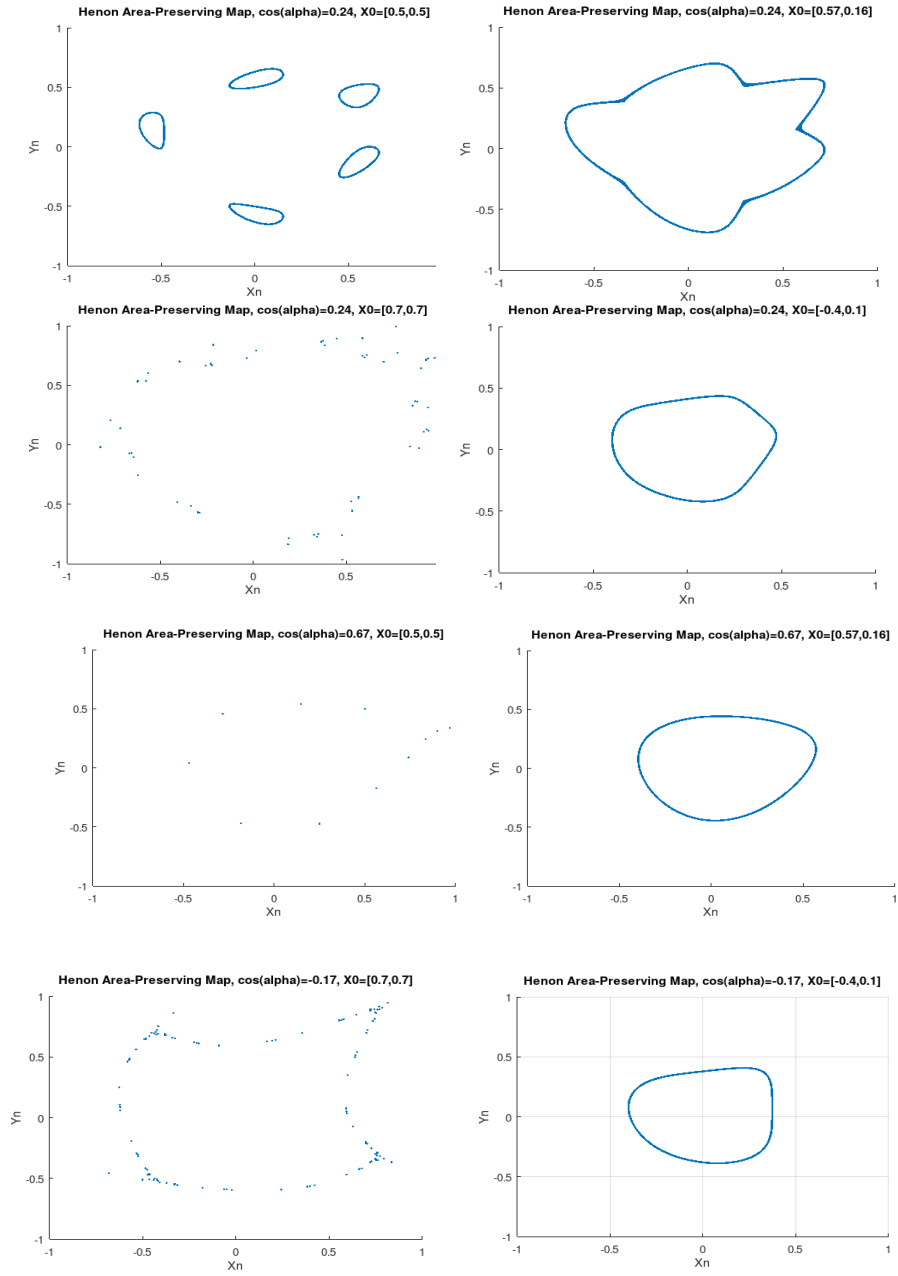


Figure 1: Exercise 6(c): Henon Area-Preserving Map Iteration for Various $\cos \alpha$ and (x_0, y_0) .

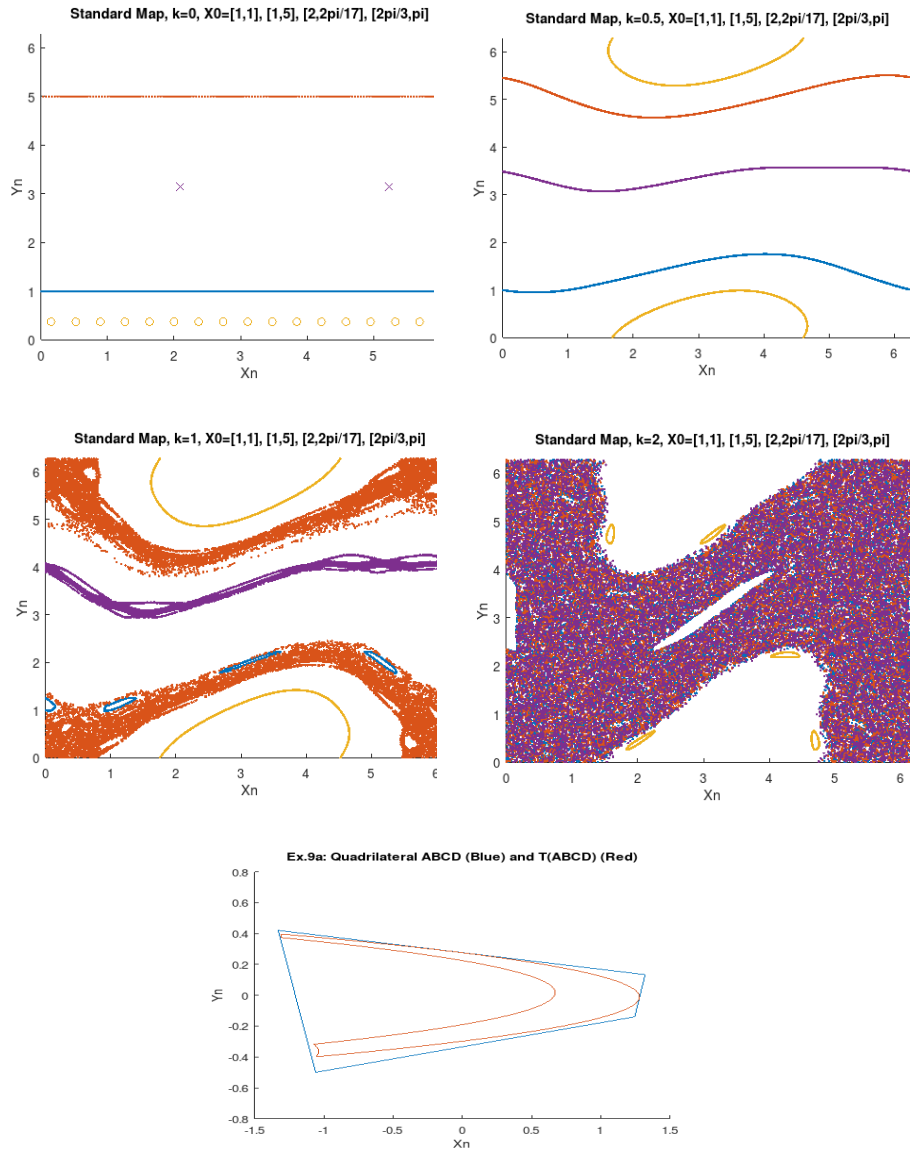


Figure 2: Top 4 plots: Ex.7b, Standard Map Orbits for Specified k and Various (x_0, y_0) . Bottom plot: Ex.9a, Image of the Trapping Region Under the Henon Mapping.