Ma 4102: Introduction to Lebesgue Integration
Homework Assignment 1

Prof. Wickerhauser

Read Chapters 1 and 2 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

1. (Ex.5.6, p.9) Let $f(x) = 1$ for $x = 1/n$, $n = 1, 2, \ldots$, and suppose $f(x) = 0$ otherwise. Show that f is Riemann integrable (R.I.) and that $\int_0^1 f = 0$.

2. (Ex.5.20,p.11) Invent a function which is monotone on $[0, 1]$ but is not piecewise continuous.

3. (Ex.5.21,p.11) Let χ_A denote the characteristic function of set A:

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

   (a) Prove that for any sets A, B, $\chi_{A\cap B} = \chi_A \chi_B$.

   (b) Find a similar expressions for $\chi_{A\cup B}$ and $\chi_{A\setminus B}$

   (c) Show that $\chi_A + \chi_B = \chi_{A\cap B} + \chi_{A\cup B}$.

4. (Ex.5.25,p.11) Suppose that $\{f_n : n = 1, 2, 3 \ldots\}$ is a sequence of R.I. functions on $[a, b]$ and that $f_n \to f$ uniformly as $n \to \infty$. Show that f is R.I. and that

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

5. (Ex.9.3,p.26) Prove that if μ is a measure on $\mathcal{S}$ and $\{x\} \in \mathcal{S}$ for every $x \in [a, b]$, and $\mu(\{x\}) = \mu(\{y\})$ for all $x, y \in [a, b]$, then $\mu(\mathbf{Q}) = 0$.

6. (Ex.9.13,p.27) Do there exist open subsets G_1, G_2 of E such that $G_1 \neq G_2$ but $\mu^*(G_1) = \mu^*(G_2)$?

7. (Ex.9.15,p.27) Prove that μ^* is countably additive on the class of open subsets of E.

8. (Ex.9.17,p.27) Show that if $A \subset B \subset E$, then $\mu^*(A) \leq \mu^*(B)$.

9. (Ex.9.21,p.27) In Example 7.7 on p.20, prove that if $E_\alpha \cap E_\beta \neq \emptyset$, then $E_\alpha = E_\beta$.

10. (Ex.9.27,p.28) Prove that if $A, B \subset E$ and $\mu^*(A) + \mu^*(B) = \mu_*(A) + \mu_*(B)$, then A and B are measurable. (Hint: Lemma 8.7, p.25.)