Ma 4102: Introduction to Lebesgue Integration
Homework Assignment 2

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Read Chapters 3 and 4 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Note: Many of the exercises in sections 16 (Ch.3) and 20 (Ch.4), in addition to those assigned, are relatively easy and worthy of at least a mental or sketched solution.

1. (Ex.16.3,p.41) Prove that if A and B are measurable subsets of E, then

   $$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$

   [Hint: Use 10.2 and 10.3.]

2. (Ex.16.5,p.41)

   (a) Using Cor.10.3, show that if A and B are measurable subsets of E, then $A \setminus B$ is measurable.

   (b) Use countable additivity (10.6) to show that if $B \subset A$ and A, B are measurable subsets of E, then $\mu^*(A \setminus B) = \mu^*(A) - \mu^*(B)$.

3. (Ex.16.14,p.42) Denote the Borel subsets of E by $\mathcal{B}$.

   (a) Show that every open subset of E is in $\mathcal{B}$.

   (b) Show that every closed subset of E is in $\mathcal{B}$.

   (c) Show that every half-open interval $(a, b] \subset E$ is in $\mathcal{B}$.

   (d) Show that $\mathbf{Q} \cap E$ is in $\mathcal{B}$.

4. (Ex.16.15,p.42) Show that if $f : E \to E$ is continuous and $A \subset E$ is a Borel set, then its preimage

   $$f^{-1}(A) \stackrel{\text{def}}{=} \{x \in E : f(x) \in A\}$$

   is a Borel set.

   [Hint: show that $\{A \subset E : f^{-1}(A) \in \mathcal{B}\}$ is a σ-algebra containing all the open subintervals of E.]

5. (Ex.16.25,p.43) Suppose that A and B are measurable subsets of E.

   (a) Use Carathéodory's criterion (Th.13.1) to prove that $A \cup B$ is measurable.

   (b) Use the squeeze criterion (Th.13.2) to prove that $A \cup B$ is measurable.

6. (Ex.16.39, p.45) Prove Thm.12.9 (as corrected here): Given any Lebesgue measurable set $A \subset E$, there exists a Borel set $B \subset E$ such that $B \subset A$ and $\mu(A) = \mu(B)$. That is, B differs from A by a set of measure zero.

7. (Ex.20.4,p.55) Prove that if $f : E \to \mathbf{R}$ is measurable, then

$$f^{-1}(c) \stackrel{\text{def}}{=} \{x \in E : f(x) = c\}$$

is measurable for every real number c.

8. (Ex.20.6,p.55) Suppose that $B \subset E$ is a set, $f : B \to \mathbf{R}$ is a function, and the set $\{x \in B : f(x) < c\}$ is measurable for each real number c. Prove that B is measurable.

9. (Ex.20.15,p.57) Show that if f is continuous a.e. on a compact set K, then for every $\epsilon > 0$ there is a measurable set $A \subset K$ such that f is bounded on A and $\mu(K \setminus A) < \epsilon$.

   [Hint: consider the sets $\{x \in K : |f(x)| < N\}$ for $N = 1, 2, \ldots$]

10. (Ex.20.20,p.57) Prove or find a counterexample: if $|f|$ is measurable, then f is measurable.