Ma 4102: Introduction to Lebesgue Integration
Homework Assignment 3

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Read Chapters 5 and 6 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**
Note: Many of the exercises in sections 26 (Ch.5) and 31 (Ch.6), in addition to those assigned, are worthy of your efforts.

1. (Ex.26.3,p.74) Show that if f is monotone on $[a,b]$ then every Riemann sum of f is a Lebesgue sum. Is the converse true?

2. (Ex.26.3,p.74) Prove Proposition 22.3: If f is simple on $A \cup B$, where A, B are bounded measurable disjoint subsets of $\mathbf{R}$, then
$$\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu.$$

3. (Ex.26.21,p.75) Prove that for any measurable function $f : A \to \mathbf{R}$, if $\mu(A) = 0$ then $\int_A f\,d\mu = 0$.

4. (Ex.26.27 and Ex.26.31, p.75) Suppose that B is a measurable subset of measurable $A \subset \mathbf{R}$ and that $f \in \mathcal{L}(A)$.
   (a) Prove that $f \in \mathcal{L}(B)$.
   (b) Prove that $\chi_B f \in \mathcal{L}(A)$ and $\int_A \chi_B f\,d\mu = \int_B f\,d\mu$.

5. (Ex.31.1,p.88) Show that if $f_n \to f$ uniformly on a bounded measurable set $A \subset \mathbf{R}$, and $(\forall n) f_n \in \mathcal{L}(A)$, then $f \in \mathcal{L}(A)$ and
$$\lim_{n\to\infty} \int_A f_n\,d\mu = \int_A f\,d\mu.$$

6. (Ex.31.5 and Ex.31.6, p.88) Use the Monotone Convergence Theorem (28.2, p.79).
   (a) Show that $f(x) = 1/x$ is not summable on $A = (0,1) \subset \mathbf{R}$.
   [Hint: consider the sequence of functions $\{f_n \stackrel{\text{def}}{=} \max(f,n) : n = 1,2,\ldots\}$.]
   (b) Find $\int_A g\,d\mu$ for $g(x) = 1/\sqrt{x}$.

7. (Ex.31.11,p.89) Show that strict inequality may hold in the conclusion of Fatou's Lemma (28.7, p.81)

8. (Ex.31.20,p.89) Let $A \subset \mathbf{R}$ be bounded and measurable. Suppose $f, h \in \mathcal{L}(A)$ with $\int_A f\,d\mu = \int_A h\,d\mu$. Show that if $f \le g \le h$ on A, then $g \in \mathcal{L}(A)$ and $\int_A g\,d\mu = \int_A f\,d\mu$.

9. (Ex.31.21,p.90) Give an example to show that Egoroff's Theorem (30.1, p.85) cannot be improved to yield $\mu(A_\epsilon) = 0$.

10. (Ex.31.25,p.90) Let $f = \chi_{\mathbf{Q}}$ on $[0,1]$, Given $\epsilon > 0$, find C_ϵ as in Lusin's Theorem (30.3, p.87).