Ma 4102: Introduction to Lebesgue Integration
Homework Assignment 4

Prof. Wickerhauser

Read Chapter 7 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**
Note: Many of the exercises in section 34 of Chapter 7, in addition to those assigned, are worthy of your efforts.

1. (Ex.34.3,p.108) Let $M \stackrel{\text{def}}{=} \{f : [0,1] \to \mathbf{R} | f$ attains both a minimum and a maximum value.$\}$ Show that M is not a vector space, namely it is not preserved by addition and scalar multiplication.

2. (Ex.34.5 and 34.18,p.109)

   (a) For any norm $\|\cdot\|$, prove that $|\|x\| - \|y\|| \leq \|x - y\|$

   (b) For the derived norm $\|x\|$, prove that $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

   (c) Find a norm for which (b) does not hold.

3. (Ex.34.8,p.109) Prove that if $f_n \in C([0,1])$ for $n = 1, 2, \ldots$, and $f_n \to f$ in the supremum norm (Example 32.4(2),p.95), then $f_n \to f$ in the $\mathcal{L}^1$ norm (Example 32.4(3),p.95).

4. (Ex.34.9,p.109) Prove that any convergent sequence in a normed linear space is a Cauchy sequence.

5. (Ex.34.21,p.110, corrected)

   (a) Prove that an orthonormal set in an inner product space is linearly independent.

   (b) Prove that if $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V, then any $v \in V$ can be expressed as

$$v = \sum_{i=1}^{n} (v \cdot v_i) v_i$$

6. (Ex.34.29 and 34.30,p.111)

   (a) Prove that if $f_n, f \in \mathcal{L}^2(A)$ and $f_n \to f$ uniformly on A, then $f_n \to f$ in $\mathcal{L}^2$.

   (b) Find a counterexample to show that pointwise convergence does not imply $\mathcal{L}^2$ convergence.

7. (Ex.34.32,p.111) Let $f_n, f, g \in \mathcal{L}^2(A)$, and suppose $f_n \to f$ in $\mathcal{L}^2$. Prove that

$$\lim_{n\to\infty} \int_A f_n g \, d\mu = \int_A f g \, d\mu.$$

8. (Ex.34.36,p.111) Show that the unit sphere $\{f \in \mathcal{L}^2([0,1]) : \|f\| = 1\}$ is not compact.

   [Hint: find a sequence $\{g_n : n = 1, 2, \ldots\}$ with $\|g\| = 1$ but with $\|g_n - g_m\| \geq 1$ for all $n \neq m$.]

9. (Ex.34.33,p.111) Prove that if $f_n \in \mathcal{L}^2(A)$ for all n, and $f_n \to f$ in $\mathcal{L}^2$, then $f \in \mathcal{L}^2(A)$.

10. (Ex.34.41,p.112) Suppose that $A \subset \mathbf{R}$ is bounded and Lebesgue measurable, and f is a measurable function defined on A.

    (a) Show that $\|f\|_1 \leq \|f\|_2 \sqrt{\mu(A)}$.

    (b) Find a sequence $\{f_n\} \subset \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$ which converges to 0 in $\mathcal{L}^1$ but does not converge in $\mathcal{L}^2$. □