Ma 4102: Introduction to Lebesgue Integration
Homework Assignment 5

Prof. Wickerhauser

Read Chapter 8 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Note: Many of the exercises in section 38 of Chapter 8, in addition to those assigned, are worthy of your efforts.

1. (Ex.38.1,p.127) A function $f \in \mathcal{L}^2([-\pi, \pi])$ is said to be *even* iff $f(-x) = f(x)$ for all $x \in [-\pi, \pi]$. It is said to be *odd* iff $f(-x) = -f(x)$ for all $x \in [-\pi, \pi]$. Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for f.

(a) Prove that if f is even, then $b_n = 0$ for all $n = 1, 2, \ldots$

(b) Prove that if f is odd, then $a_n = 0$ for all $n = 0, 1, 2, \ldots$

2. (Ex.38.9,p.128) Suppose that $\sum_{k=1}^{\infty} a_k$ is an infinite series with $a_k \geq 0$ for every $k = 1, 2, \ldots$. Let σ_n be the arithmetic mean of its first n partial sums as in Def.37.1, p.121.

(a) Prove that $\sigma_n \leq \sigma_{n+1}$ for every $n = 1, 2, \ldots$

(b) Prove that if $\sum_{k=1}^{\infty} a_k = \infty$, then $\lim_{n\to\infty} \sigma_n = \infty$.

[Conclude that a series with non-negative terms converges to a real number if and only if it $(C, 1)$ converges to that number.]

3. (Ex.38.14,p.129) If $f \in \mathcal{L}([-\pi, \pi])$ is extended periodically to $\mathbf{R}$, show that for any $x \in \mathbf{R}$,

$$\int_{[x-\pi,x+\pi]} f\, d\mu = \int_{[-\pi,\pi]} f\, d\mu$$

[HINT: draw a picture.]

4. (Ex.38.16,p.129) Prove the following analogue to Lem.37.7, p.125:

(a) $\displaystyle\int_0^{\pi} D_n(t)\, dt = \pi/2$, and

(b) for $0 < \delta \leq |t| \leq \pi$, we have the inequality $|D_n(t)| \leq \dfrac{1}{2|\sin(\delta/2)|}$.

5. (Ex.38.18,p.129) Prove the following:

   (a) If f is continuous on $[-\pi, \pi]$ and its Fourier coefficients are all 0, then $f \equiv 0$.

   (b) If f and g are continuous on $[-\pi, \pi]$ and the Fourier coefficients of f and g are identical, then $f(x) = g(x)$ for all x.

   (c) If f and g belong to $\mathcal{L}^2([-\pi, \pi])$ and have identical Fourier coefficients, then $f(x) = g(x)$ a.e.$x \in [-\pi, \pi]$.

6. (Ex.38.20,p.129) Prove Parseval's formula: if $f \in \mathcal{L}^2([-\pi, \pi])$, then

$$\frac{1}{\pi} \|f\|_2^2 \;=\; \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$

   [HINT: compute $\|s_n - f\|_2^2$ and apply Th.37.9.]

7. (Ex.38.22,p.130) Suppose that $f, g \in \mathcal{L}^2([-\pi, \pi])$ have respective Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \qquad \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx).$$

   Use Parseval's formula with $\|f + g\|_2^2$ and $\|f - g\|_2^2$ to show that

$$\frac{1}{\pi}\,(f \cdot g) = \frac{a_0 \alpha_0}{2} + \sum_{k=1}^{\infty}(a_k \alpha_k + b_k \beta_k)$$

   [This is a major application of the "polarization identity."]

8. (Ex.38.27,p.130) If f is 2π-periodic and continuously differentiable on $[-\pi, \pi]$, show that the Fourier transform of f' can be obtained from that of f by differentiating term-by-term.

   [HINT: integrate by parts.]

9. (Ex.38.29,p.131) Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are absolutely convergent series. Show that

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

   converges absolutely and uniformly on $[-\pi, \pi]$ to a continuous function.

   [HINT: Use the Weierstrass M test.]

10. (Ex.38.30,p.131) Show that if f is continuous on $[-\pi, \pi]$ and the Fourier transform of f converges at x, then it converges to $f(x)$ at x.

    [HINT: apply Th.37.8, p.125.]