Ma 4102: Introduction to Lebesgue Integration
Solutions to Homework Assignment 1

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Read Chapters 1 and 2 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

1. (Ex.5.6, p.9) Let $f(x) = 1$ for $x = 1/n$, $n = 1, 2, \ldots$, and suppose $f(x) = 0$ otherwise. Show that f is Riemann integrable (R.I.) and that $\int_0^1 f = 0$.

   Solution: Let $\epsilon > 0$ be given. Let $f_1 = 0$ (i.e., the zero step function) and let f_2 be the step function defined by

   $$f_2(x) = \begin{cases} 1, & 0 \le x < \epsilon \text{ or } |x - 1/n| < \epsilon/2^n, \\ 0, & \text{otherwise.} \end{cases}$$

   Then $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [0, 1]$, and

   $$\left| \int_0^1 f_2 - \int_0^1 f_1 \right| = \left| \int_0^1 f_2 - 0 \right| = \left| \int_0^\epsilon f_2 + \int_\epsilon^1 f_2 \right| \le \epsilon + 2 \sum_{n=1}^\infty \frac{\epsilon}{2^n} \;=\; 3\epsilon.$$

   Conclude by Theorem 1.6 (p.5) that f is R.I. and that $\int_0^1 f = 0$. □

2. (Ex.5.20,p.11) Invent a function which is monotone on $[0, 1]$ but is not piecewise continuous.

   Solution: One example: $f(x) = 1/n$ for $\frac{1}{n+1} < x \le \frac{1}{n}$, with $f(0) = 0$. This f is monotone, in fact increasing, but is not piecewise continuous because it has infinitely many points of discontinuity. □

3. (Ex.5.21,p.11) Let χ_A denote the characteristic function of set A:

   $$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

   (a) Prove that for any sets A, B, $\chi_{A\cap B} = \chi_A \chi_B$.

   (b) Find a similar expressions for $\chi_{A\cup B}$ and $\chi_{A\setminus B}$

   (c) Show that $\chi_A + \chi_B = \chi_{A\cap B} + \chi_{A\cup B}$.

   Solution:

   (a) $\chi_{A\cap B}(x) = 1$ iff $x \in A$ and $x \in B$, iff $\chi_A(x) = \chi_B(x) = 1$ giving $\chi_A(x)\chi_B(x) = 1$, otherwise the product is zero.

   (b) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$, while $\chi_{A\setminus B} = \chi_A - \chi_{A\cap B}$.

   (c) This follows immediately from part b. □

4. (Ex.5.25,p.11) Suppose that $\{f_n : n = 1, 2, 3 \ldots\}$ is a sequence of R.I. functions on $[a, b]$ and that $f_n \to f$ uniformly as $n \to \infty$. Show that f is R.I. and that

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Solution: Let $R_n = \int_a^b f_n$ and observe that $R \stackrel{\text{def}}{=} \lim_{n\to\infty} R_n$ exists because $\{R_n\}$ is a Cauchy sequence:

$$|R_n - R_m| = \left| \int_a^b f_n - \int_a^b f_m \right| = \left| \int_a^b (f_n - f_m) \right| = \left| \int_a^b (f_n - f + f - f_m) \right| \leq 2\epsilon(b - a),$$

whenever n and m are large enough to insure $|f_n(x) - f(x)| < \epsilon$ and $|f_m(x) - f(x)| < \epsilon$ for all $x \in [a, b]$.

Now let $\epsilon > 0$ be given and find n such that $(\forall x)|f_n(x) - f(x)| < \epsilon$. By Theorem 1.6, since f_n is R.I., there exist step functions U_n and L_n satisfying $L_n \leq f_n \leq U_n$ and $\int_a^b U_n - \int_a^b L_n < \epsilon$. Modify these into step functions $U \stackrel{\text{def}}{=} U_n + \epsilon$ and $L \stackrel{\text{def}}{=} L_n - \epsilon$ and note that these satisfy

$$(\forall x \in [a, b])L(x) \leq f(x) \leq U(x); \qquad \left| \int_a^b U - \int_a^b L \right| < \epsilon + 2\epsilon|b - a| = (1 + 2|b - a|)\epsilon.$$

Conclude by Theorem 1.6 that f is R.I.

Finally, observe that

$$\left| R - \int_a^b f \right| = \left| R - R_n + \int_a^b f_n - \int_a^b f \right| \leq |R - R_n| + \left| \int_a^b f_n - \int_a^b f \right| < |R - R_n| + \epsilon|b - a|,$$

and thus conclude, by letting $n \to \infty$, that $\int_a^b f = R$ as claimed. □

5. (Ex.9.3,p.26) Prove that if μ is a measure on $\mathcal{S}$ and $\{x\} \in \mathcal{S}$ for every $x \in [a, b]$, and $\mu(\{x\}) = \mu(\{y\})$ for all $x, y \in [a, b]$, then $\mu(\mathbf{Q} \cap [a, b]) = 0$.

Solution: First note that $[a, b]$ contains infinitely many points but (by Def.6.1.ii) $\mu([a, b]) \leq b - a < \infty$, forcing $\mu(\{x\}) = 0$ for all $x \in [a, b]$ by the hypotheses and monotonicity.

Since every subset of countable $\mathbf{Q}$ is countable, we may denumerate

$$\mathbf{Q} \cap [a, b] = \{q_1, q_2, \ldots\} = \bigcup_{i=1}^{\infty} \{q_i\},$$

where the union is disjoint and $\{q_i\} \in \mathcal{S}$ for every $i = 1, 2, \ldots$, so $\mathbf{Q} \cap [a, b] \in \mathcal{S}$ by its closure under countable unions. Therefore, by the countable additivity of measure μ,

$$\mu(\mathbf{Q} \cap [a, b]) = \mu\left(\bigcup_{i=1}^{\infty} \{q_i\}\right) = \sum_{i=1}^{\infty} \mu(\{q_i\}) = \sum_{i=1}^{\infty} 0 = 0.$$

□

6. (Ex.9.13,p.27) Do there exist open subsets G_1, G_2 of E such that $G_1 \neq G_2$ but $\mu^*(G_1) = \mu^*(G_2)$?

Solution: Yes. One example is $G_1 = (0, 0.5) \neq G_2 = (0.5, 1)$, with $\mu^*(G_1) = \mu^*(G_2) = 0.5$. □

7. (Ex.9.15,p.27) Prove that μ^* is countably additive on the class of open subsets of E.

Solution: This implies that open sets are Lebesgue measurable.

Suppose that $G = \cup_{n=1}^{\infty} G_n$ is a countable union of disjoint open subsets of E. Using Th.7.2, write each G_n as a unique disjoint union of open intervals $I_{ni}, i = 1, 2, \ldots$:

$$G_n = \bigcup_{i=1}^{\infty} I_{ni}, \qquad n = 1, 2, \ldots$$

Apply Def.7.3 to compute

$$\mu^*(G_n) = \sum_i \mu^*(I_{ni})$$

Since $n \neq m$ implies $G_n \cap G_m = \emptyset$, and $i \neq j \Rightarrow I_{ni} \cap I_{nj} = \emptyset$ for all n, it follows that

$$(n, i) \neq (m, j) \Rightarrow I_{ni} \cap I_{mj} = \emptyset.$$

Also, the union of disjoint nonempty open intervals cannot be a single interval, so the double union is again a (unique, disjoint, countable) decomposition into component intervals of G:

$$G = \bigcup_n \bigcup_i I_{ni}$$

Apply Def.7.3 to this double decomposition to get

$$\mu^*(G) = \mu^*\left(\cup_n G_n\right) = \mu^*\left(\cup_n \cup_i I_{ni}\right) = \sum_n \sum_i \mu^*\left(I_{ni}\right) = \sum_n \mu^*\left(\cup_i I_{ni}\right) = \sum_n \mu^*\left(G_n\right),$$

where we may sum in any order since all summands are nonnegative. □

8. (Ex.9.17,p.27) Show that if $A \subset B \subset E$, then $\mu^*(A) \leq \mu^*(B)$.

Solution: This follows from Def.7.4. Since $A \subset B \subset E$, any open set $G \subset E$ with $B \subset G$ also satisfies $A \subset G$, so

$$\{G^{\text{open}} \subset E : B \subset G\} \subset \{G^{\text{open}} \subset E : A \subset G\},$$

so

$$\mu^*(B) = \inf\{\mu^*(G) : B \subset G\} \ \geq \ \inf\{\mu^*(G) : A \subset G\} = \mu^*(A),$$

since the infimum on the right is taken over a larger set of open covers. □

9. (Ex.9.21,p.27) In Example 7.7 on p.20, prove that if $E_\alpha \cap E_\beta \neq \emptyset$, then $E_\alpha = E_\beta$.

Solution: Suppose that $x \in E_\alpha \cap E_\beta \neq \emptyset$. Then $x - \alpha \in \mathbf{Q}$ and $x - \beta \in \mathbf{Q}$, so

$$\alpha - \beta = (x - \beta) - (x - \alpha) \in \mathbf{Q}.$$

Thus any $y \in E_\alpha$ with $y - \alpha \in \mathbf{Q}$ also satisfies

$$y - \beta = y - \alpha + (\alpha - \beta) \in \mathbf{Q}, \quad \Rightarrow y \in E_\beta$$

from which we conclude that $E_\alpha \subset E_\beta$. Interchange α, β in the prior argument to see that $E_\beta \subset E_\alpha$ and conclude that $E_\alpha = E_\beta$. □

10. (Ex.9.27,p.28) Prove that if $A, B \subset E$ and $\mu^*(A) + \mu^*(B) = \mu_*(A) + \mu_*(B)$, then A and B are measurable. (Hint: Lemma 8.7, p.25.)

Solution: By Lem.8.7, $\mu^*(A) \geq \mu_*(A)$. Subtracting this inequality from $\mu^*(A) + \mu^*(B) = \mu_*(A) + \mu_*(B)$ gives

$$\mu^*(B) \leq \mu_*(B).$$

which with $\mu^*(B) \geq \mu_*(B)$ (from Lem.8.7 applied to B) gives $\mu^*(B) = \mu_*(B)$. Conclude that B is measurable.

Interchange B and A in the prior argument to conclude that A is measurable. □