Ma 4102: Introduction to Lebesgue Integration
Solutions to Homework Assignment 2

Prof. Wickerhauser

Read Chapters 3 and 4 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Note: Many of the exercises in sections 16 (Ch.3) and 20 (Ch.4), in addition to those assigned, are relatively easy and worthy of at least a mental or sketched solution.

1. (Ex.16.3,p.41) Prove that if A and B are measurable subsets of E, then

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$

[Hint: Use 10.2 and 10.3.]

Solution: By Corollary 10.3, if A and B are measurable then so are $A \cup B$ and $A \cap B$. Thus by definition $\mu^*(A \cup B) = \mu_*(A \cup B)$ and $\mu^*(A \cap B) = \mu_*(A \cap B)$ as well as $\mu^*(A) = \mu_*(A)$ and $\mu^*(B) = \mu_*(B)$. Substituting these into Lemma 10.2(2) gives

$$\mu^*(A) + \mu^*(B) \leq \mu^*(A \cup B) + \mu^*(A \cap B),$$

which together with Lemma 10.2(1),

$$\mu^*(A) + \mu^*(B) \geq \mu^*(A \cup B) + \mu^*(A \cap B),$$

implies the result. □

2. (Ex.16.5,p.41)

(a) Using Cor.10.3, show that if A and B are measurable subsets of E, then $A \setminus B$ is measurable.

(b) Use countable additivity (10.6) to show that if $B \subset A$ and A, B are measurable subsets of E, then $\mu^*(A \setminus B) = \mu^*(A) - \mu^*(B)$.

Solution: (a) A is measurable by hypothesis and $E \setminus B$ is measurable by Corollary 8.4, so $A \setminus B = A \cap (E \setminus B)$ is the intersection of measurable sets and is thus measurable by Corollary 10.3.

(b) Expand $A = (A \cap B) \cup (A \cap (E \setminus B)) = (A \cap B) \cup (A \setminus B)$, which is a disjoint union. Thus by countable additivity,

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B).$$

But also, since $B \subset A$, we have $A \cap B = B$, so

$$\mu^*(A) = \mu^*(B) + \mu^*(A \setminus B).$$

All quantities are finite, so the result follows by subtraction. □

3. (Ex.16.14,p.42) Denote the Borel subsets of E by $\mathcal{B}$.

   (a) Show that every open subset of E is in $\mathcal{B}$.

   (b) Show that every closed subset of E is in $\mathcal{B}$.

   (c) Show that every half-open interval $(a, b] \subset E$ is in $\mathcal{B}$.

   (d) Show that $\mathbf{Q} \cap E$ is in $\mathcal{B}$.

   Solution:

   (a) Let $G \subset E$ be open. By Th.7.2, $G = \cup_k I_k$ is a countable union of open intervals $\{I_k\}$. But $I_k \in \mathcal{B}$ for all k by definition, and $\mathcal{B}$ is closed under countable unions, so $G \in \mathcal{B}$.

   (b) By definition, $F \subset E$ is closed iff $F = E \setminus G$ for some open set $G \subset E$. By part (a), $G \in \mathcal{B}$. But $\mathcal{B}$ is closed under set complementation, so $F = E \setminus G \in \mathcal{B}$.

   (c) Write $(a, b] = [0, b] \cap (a, 1]$ and note that $[0, b] \subset E$ is closed while $(a, 1] \subset E$ is (relatively) open, hence both are Borel sets. Conclude that their intersection is also in $\mathcal{B}$, which is closed under countable intersections.

   (d) For every $x \in E$, the singleton $\{x\} \in \mathcal{B}$ since it is the countable intesection of the intervals

   $$G_k = (x - 2^{-k}, x + 2^{-k}) \cap E, \qquad k = 1, 2, \ldots$$

   which are relatively open in E and hence Borel sets. Denumerate the countable set $\mathbf{Q} \cap E = \{q_1 . q_2, \ldots\}$ and observe that

   $$\mathbf{Q} \cap E = \bigcup_{n=1}^{\infty} \{q_n\}$$

   is a countable union of Borel sets, hence it is a Borel set.

   □

4. (Ex.16.15,p.42) Show that if $f : E \to E$ is continuous and $A \subset E$ is a Borel set, then its preimage

   $$f^{-1}(A) \stackrel{\text{def}}{=} \{x \in E : f(x) \in A\}$$

   is a Borel set.

   [Hint: show that $\{A \subset E : f^{-1}(A) \in \mathcal{B}\}$ is a σ-algebra containing all the open subintervals of E.]

   Solution: Following the hint, put

   $$\mathcal{A} \stackrel{\text{def}}{=} \{A \subset E : f^{-1}(A) \in \mathcal{B}\}.$$

   Then $\emptyset \in \mathcal{A}$ since $f^{-1}(\emptyset) = \emptyset \in \mathcal{B}$. Also $E \in \mathcal{A}$ since $f^{-1}(E) = E \in \mathcal{B}$. Likewise,

   $$\{A_k : k = 1, 2, \ldots\} \subset \mathcal{A} \Rightarrow f^{-1}(\cup_k A_k) = \cup_k f^{-1}(A_k) \in \mathcal{B}, \quad \Rightarrow \cup_k A_k \in \mathcal{A},$$

   since $\mathcal{B}$ is preserved by countable unions and $f^{-1}(A_k) \in \mathcal{B}$ for each $k = 1, 2, \ldots$ Conclude that $\mathcal{A}$ is preserved by countable unions. Combine this with closure under complements to conclude that $\mathcal{A}$ is closed under countable intersections as well. Thus $\mathcal{A}$ is a σ algebra.

   Now let $I \subset E$ be an open interval. Since f is continuous, $f^{-1}(I)$ is an open set, so $f^{-1}(I) \in \mathcal{B}$ by the remarks after Def.12.4, so $I \in \mathcal{A}$. Hence $\mathcal{A}$ is a σ-algebra containing all open intervals $I \subset E$.

   But the Borel σ-algebra $\mathcal{B}$ is the intersection of all such σ-algebras, so $\mathcal{B} \subset \mathcal{A}$. Therefore,

   $$A \in \mathcal{B} \subset \mathcal{A} \ \Rightarrow f^{-1}(A) \in \mathcal{B},$$

   as claimed.

   □

5. (Ex.16.25,p.43) Suppose that A and B are measurable subsets of E.

   (a) Use Carathéodory's criterion (Th.13.1) to prove that $A \cup B$ is measurable.

   (b) Use the squeeze criterion (Th.13.2) to prove that $A \cup B$ is measurable.

Solution: (a) This proof uses set algebra identities. For brevity, use the notation $A^c \stackrel{\text{def}}{=} E \setminus A$ for subsets $A \subset E$.

Let $X \subset E$ be arbitrary.

First, apply Carathéodory's criterion with measurable A to the arbitrary set $X \cap (A \cup B)$ to get

$$\begin{aligned} \mu^*(X \cap (A \cup B))) &= \mu^*(X \cap (A \cup B) \cap A) + \mu^*(X \cap (A \cup B) \cap A^c) \\ &= \mu^*(X \cap A) + \mu^*(X \cap A^c \cap B), \end{aligned}$$

since $(A \cup B) \cap A = A$ and $(A \cup B) \cap A^c = A^c \cap B$.

Second, use DeMorgan's identity $(A \cup B)^c = A^c \cap B^c$ to compute

$$\mu^*(X \cap (A \cup B)^c) = \mu^*(X \cap A^c \cap B^c).$$

Combine these two expansions to get

$$\begin{aligned} \mu^*(X \cap (A \cup B)) + \mu^*(X \cap (A \cup B)^c) &= \mu^*(X \cap A) + \mu^*(X \cap A^c \cap B) + \mu^*(X \cap A^c \cap B^c) \\ &= \mu^*(X \cap A) + \mu^*(X \cap A^c) \\ &= \mu^*(X), \end{aligned}$$

where the second step relies on Carathéodory's criterion with measurable B applied to the arbitrary set $X \cap A^c$, and the last step reuses Carathéodory's criterion with measurable A. Conclude that Carathéodory's criterion applies to $A \cup B$.

(b) Suppose that A and B are measurable. Let $\epsilon > 0$ be given.

By Th.13.2, there exist open sets G_A, H_A and G_B, H_B such that

$$A \subset G_A, \quad A^c \subset H_A, \quad B \subset G_B, \quad B^c \subset H_B,$$

and also satisfying $\mu(G_A \cap H_A) < \epsilon/2$ and $\mu(G_B \cap H_B) < \epsilon/2$.

But $A \cup B \subset G_A \cup G_B \stackrel{\text{def}}{=} G$, while

$$(A \cup B)^c = A^c \cap B^c \subset H_A \cap H_B \stackrel{\text{def}}{=} H,$$

using DeMorgan's identity. Finite unions and intersections of open sets are open, so

$$\begin{aligned} \mu(G \cap H) &= \mu((G_A \cup G_B) \cap H_A \cap H_B) \\ &= \mu((G_A \cap H_A \cap H_B) \cup (G_B \cap H_B \cap H_A)) \qquad \text{(by set algebra)} \\ &\leq \mu(G_A \cap H_A \cap H_B) + \mu(G_B \cap H_B \cap H_A) \qquad \text{(by subadditivity)} \\ &\leq \mu(G_A \cap H_A) + \mu(G_B \cap H_B) \qquad \text{(by monotonicity)} \\ &< \epsilon \end{aligned}$$

Conclude that the squeeze criterion applies to $A \cup B$. □

6. (Ex.16.39, p.45) Prove Thm.12.9 (as corrected here): Given any Lebesgue measurable set $A \subset E$, there exists a Borel set $B \subset E$ such that $B \subset A$ and $\mu(A) = \mu(B)$. That is, B differs from A by a set of measure zero.

Solution: Let measurable $A \subset E$ be given. For each $n = 1, 2, \ldots$, apply Corollary.13.3 to choose open $G_n \subset E$ and closed $F_n \subset E$ such that $F_n \subset A \subset G_n$ and $\mu(G_n \setminus F_n) < 1/n$.

Now let

$$B \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} F_n \in \mathcal{B},$$

which is a Borel set since closed sets are Borel sets and their countable union is thus also a Borel set.

It is clear by this construction that $B \subset A$. It remains to show that $\mu(B) = \mu(A)$.

But $\mu(B) \leq \mu(A)$ by monotonicity. For the reverse inequality, note that for any positive integer n,

$$\mu(A) \leq \mu(G_n) = \mu(F_n) + \mu(G_n \setminus F_n) \leq \mu(B) + \mu(G_n \setminus F_n) < \mu(B) + \frac{1}{n},$$

also by monotonicity since $A \subset G_N$ and $F_n \subset B$ for all n. Conclude that $\mu(A) \leq \mu(B)$, and so $\mu(B) = \mu(A)$. □

7. (Ex.20.4,p.55) Prove that if $f : E \to \mathbf{R}$ is measurable, then

$$f^{-1}(c) \stackrel{\text{def}}{=} \{x \in E : f(x) = c\}$$

is measurable for every real number c.

Solution: Let c be given. For $n = 1, 2, \ldots$, let

$$B_n \stackrel{\text{def}}{=} \{x \in E : c - \frac{1}{n} < f(x) < c + \frac{1}{n}\} = f^{-1}\left((c - \frac{1}{n}, c + \frac{1}{n})\right)$$

Then B_n is measurable by Prop.17.2(8) for all n, and so the countable intersection $B \stackrel{\text{def}}{=} \cap_n B_n$ is measurable. It remains to compute

$$f^{-1}(\{c\}) = f^{-1}\left(\bigcap_n (c - \frac{1}{n}, c + \frac{1}{n})\right) = \bigcap_n f^{-1}\left((c - \frac{1}{n}, c + \frac{1}{n})\right) = \bigcap_n B_n = B.$$

Conclude that $f^{-1}(c)$ is measurable. □

8. (Ex.20.6,p.55) Suppose that $B \subset E$ is a set, $f : B \to \mathbf{R}$ is a function, and the set $\{x \in B : f(x) < c\}$ is measurable for each real number c. Prove that B is measurable.

Solution: For every positive integer n, let

$$B_n \stackrel{\text{def}}{=} \{x \in B : f(x) < n\}.$$

Then $B_n \subset B$ for every n, so $\cup_n B_n \subset B$. But also $B \subset \cup_n B_n$ since every $x \in B$ belongs to B_n for some n (function f takes finite values on its domain B). Conclude that $B = \cup_n B_n$, which as the countable union of measurable sets must be measurable. □

9. (Ex.20.15,p.57) Show that if f is continuous a.e. on a compact set K, then for every $\epsilon > 0$ there is a measurable set $A \subset K$ such that f is bounded on A and $\mu(K \setminus A) < \epsilon$.

   [Hint: consider the sets $\{x \in K : |f(x)| < N\}$ for $N = 1, 2, \ldots$]

   Solution: First let $D \subset K$ be all points of discontinuity for f. Then $\mu(D) = \mu^*(D) = 0$, as sets of outer measure 0 are measurable.

   Second, let $\epsilon > 0$ be given. For each $N = 1, 2, \ldots$, choose an open set D_N such that $D \subset D_N$ and

$$\mu(D_N) < \mu(D) + \frac{\epsilon}{2^N} = \frac{\epsilon}{2^N},$$

   using Prop.7.5. Next, following the hint, let

$$B_N \stackrel{\text{def}}{=} \{x \in K : |f(x)| < N\}, \qquad B'_N \stackrel{\text{def}}{=} \{x \in K \setminus D : |f(x)| < N\},$$

   for $N = 1, 2, \ldots$. Then $B_N \subset B'_N \cup D \subset B'_N \cup D_N$, and B'_N is an open set since it is the preimage of an open set under a continuous map ($|f|$ restricted to the points of continuity $K \setminus D$ of f), so $B'_N \cup D_N$ is an open set.

   Now observe that $K \subset \bigcup_{N=1}^{\infty}(B'_N \cup D_N)$ is an open covering of the compact set K, so it has a finite subcover with maximum index $M < \infty$. Let

$$A \stackrel{\text{def}}{=} \bigcup_{N=1}^{M} B'_N,$$

   which is open hence measurable, and observe that $x \in A \Rightarrow |f(x)| < M$, so f is bounded (by M) on $A \subset K$.

   Finally, note that $K \setminus A \subset \bigcup_{N=1}^{M} D_N$, so

$$\mu(K \setminus A) \leq \sum_{N=1}^{M} \mu(D_N) < \sum_{N=1}^{\infty} \frac{\epsilon}{2^N} = \epsilon,$$

   using the monotonicity and countable subadditivity of measure μ. □

10. (Ex.20.20,p.57) Prove or find a counterexample: if $|f|$ is measurable, then f is measurable.

   Solution: The statement is false by the following counterexample: Let $V \subset E$ be the nonmeasurable set constructed in Example 7.7. Let $f : E \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 1, & x \in V \\ -1, & x \in E \setminus V. \end{cases}$$

   Then $|f| = \chi_E$ is measurable since interval E is measurable, but f is not since $V = \{x \in E : f(x) > 0\}$ is not measurable. □