Ma 4102: Introduction to Lebesgue Integration
Solutions to Homework Assignment 3

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Read Chapters 5 and 6 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Note: Many of the exercises in sections 26 (Ch.5) and 31 (Ch.6), in addition to those assigned, are worthy of your efforts.

1. (Ex.26.3,p.74) Show that if f is monotone on $[a, b]$ then every Riemann sum of f is a Lebesgue sum. Is the converse true?

   Solution: Let $P = (a = x_0 < \cdots < x_n = b)$ be a partition of $[a, b]$ and let $\{x_i^* \in [x_{i-1}, x_i] : i = 1, \ldots, n\}$ be some choice of subinterval abscissas. Since f is monotone, $y_i^* \stackrel{\text{def}}{=} f(x_i^*)$ lies between $y_{i-1} \stackrel{\text{def}}{=} f(x_{i-1})$ and $y_i \stackrel{\text{def}}{=} f(x_i)$, and

   $$A_i \stackrel{\text{def}}{=} \{x \in [a, b] : f(x) \text{ lies between } y_{i-1} \text{ and } y_i\}$$

   is an interval with $\mu(A_i) = x_i - x_{i-1}$. But then the Riemann sum $R(f, P)$ with abscissas $\{x_i^*\}$ gives

   $$R(f, P) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} y_i^* \mu(A_i) = L(f, Y),$$

   the Lebesgue sum with partition $Y \stackrel{\text{def}}{=} \{y_i : i = 1, \ldots, n\}$ and ordinates $\{y_i^*\}$.

   The converse is false. The function

   $$f(x) = \begin{cases} 1 + x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$$

   on $[0, 2]$ is not monotone but it is 1-1, so every Riemann sum will give a Lebesgue sum by the previous construction. □

2. (Ex.26.3,p.74) Prove Proposition 22.3: If f is simple on $A \cup B$, where A, B are bounded measurable disjoint subsets of $\mathbf{R}$, then

   $$\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu.$$

   Solution: Write $f = \sum_{i=1}^{n} c_i \chi_{C_i}$ for real numbers $\{c_1, \ldots, c_n\}$ and bounded measurable disjoint subsets $\{C_1, \ldots, C_n\}$ of $A \cup B$.

   Put $A_i \stackrel{\text{def}}{=} C_i \cap A$ and $B_i \stackrel{\text{def}}{=} C_i \cap B$. These are measurable for $i = 1, \ldots, n$, since intersections of measurable sets are measurable. Also, $C_i = A_i \cup B_i$ is a disjoint union for each i, so $\chi_{C_i} = \chi_{A_i} + \chi_{B_i}$, and $\mu(C_i) = \mu(A_i) + \mu(B_i)$.

Now expand f

$$f = \sum_{i=1}^{n} c_i \chi_{C_i} = \sum_{i=1}^{n} c_i(\chi_{A_i} + \chi_{B_i}) = \sum_{i=1}^{n} c_i \chi_{A_i} \; + \; \sum_{i=1}^{n} c_i \chi_{B_i},$$

which is evidently the sum of two simple functions. Each is Lebesgue integrable by the formula in Prop.22.1:

$$\int_{A\cup B} f\,d\mu = \sum_{i=1}^{n} c_i\mu(C_i) = \sum_{i=1}^{n} c_i[\mu(A_i) + \mu(B_i)] = \sum_{i=1}^{n} c_i\mu(A_i) + \sum_{i=1}^{n} c_i\mu(B_i) = \int_A f\,d\mu + \int_B f\,d\mu,$$

recognizing the last two sums as the Lebesgue integrals of f on A and B, respectively. □

3. (Ex.26.21,p.75) Prove that for any measurable function $f : A \to \mathbf{R}$, if $\mu(A) = 0$ then $\int_A f\,d\mu = 0$.

Solution: Use Cor.23.4 with Th.24.3.

First note that if f is bounded and measurable on A, and $\mu(A) = 0$, then

$$0 \le \left|\int_A f\,d\mu\right| \le | \int_A |f|\,d\mu \le M\mu(A) = 0,$$

where $|f| \le M < \infty$ is any bound for f. This is Cor.23.4.

Now suppose that f is measurable but not necessarily bounded. Let

$$A_n \overset{\text{def}}{=} \{x \in A|\; n-1 \le |f(x)| < n\}, \qquad n = 1, 2, \ldots.$$

Then $A_n \subset A$ so $\mu(A_n) = 0$, so by the first part,

$$\int_{A_n} f\,d\mu = 0, \qquad \text{for every } n = 1, 2, \ldots.$$

Now $A = \cup_{n=1}^{\infty} A_n$ is a disjoint union of measurable sets, and f is bounded on each of them, so by Th.24.3,

$$\int_A f\,d\mu = \int_{\cup_n A_n} f\,d\mu = \sum_n \int_{A_n} f\,d\mu = 0.$$

□

4. (Ex.26.27 and Ex.26.31, p.75) Suppose that B is a measurable subset of measurable $A \subset \mathbf{R}$ and that $f \in \mathcal{L}(A)$.

(a) Prove that $f \in \mathcal{L}(B)$.

(b) Prove that $\chi_B f \in \mathcal{L}(A)$ and $\int_A \chi_B f\,d\mu = \int_B f\,d\mu$.

Solution: (a) Write $A = B \cup (A \setminus B)$, a disjoint union of measurable sets, and apply Th.25.3 to nonnegative f_+ and f_- to obtain

$$\int_B f_+\,d\mu \le \int_A f_+\,d\mu < \infty; \qquad \int_B f_-\,d\mu \le \int_A f_-\,d\mu < \infty.$$

Conclude from the definition that $f \in \mathcal{L}(B)$.

(b) Since B is measurable, the simple function χ_B is measurable. Let $g \le f$ be any simple function and note that $\chi_B g \le \chi_B f$ with simple function $\chi_B g$. Likewise, any simple function $g \le \chi_B f$ satisfies $g(x) = \chi_B(x) g(x)$ for all $x \in B$ and thus may be written as $\chi_B g$.

These identities apply to $f_\pm$ with $g_\pm$, so using Cor.23.3 gives

$$\begin{aligned} \int_A \chi_B f_\pm \, d\mu &= \sup\{\int_A \chi_B g_\pm \, d\mu : \ g_\pm \text{ simple, } g_\pm \le f_\pm\} \\ &= \sup\{\int_B g_\pm \, d\mu : \ g_\pm \text{ simple, } g_\pm \le f_\pm\} \ = \ \int_B f_\pm \, d\mu, \end{aligned}$$

and the result follows by linearity from $f = f_+ - f_-$. □

5. (Ex.31.1,p.88) Show that if $f_n \to f$ uniformly on a bounded measurable set $A \subset \mathbf{R}$, and $(\forall n) f_n \in \mathcal{L}(A)$, then $f \in \mathcal{L}(A)$ and

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A f \, d\mu.$$

Solution: For each $n = 1, 2, \ldots$, $f_n \in \mathcal{L}(A)$ means f_n is measurable on A.

Since $f_n \to f$ uniformly we have $f_n \to f$ pointwise on A, so f is measurable by Th.18.6.

By uniform convergence, there exists $N < \infty$ such that $|f_n(x) - f(x)| < 1$ for all $x \in A$ and all $n \ge N$. But then $0 \le f_+ \le |f_N| + 1$ and $0 \le f_- \le |f_N| + 1$, so both $\int f_+$ and $\int f_-$ are finite, bounded above by $\mu(A) + \int |f_N| < \infty$. Conclude that $f \in \mathcal{L}(A)$.

But also, $|f_n| \le |f| + 1$ for all $n \ge N$, and $|f| + 1 \in \mathcal{L}(A)$, so by Th.28.9(DCT) it follows that

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A f \, d\mu.$$

□

6. (Ex.31.5 and Ex.31.6, p.88) Use the Monotone Convergence Theorem (28.2, p.79).

(a) Show that $f(x) = 1/x$ is not summable on $A = (0, 1) \subset \mathbf{R}$.

[Hint: consider the sequence of functions $\{f_n \stackrel{\text{def}}{=} \min(f, n) : n = 1, 2, \ldots\}$.]

(b) Find $\int_A g \, d\mu$ for $g(x) = 1/\sqrt{x}$.

Solution: (a) Follow the (corrected) hint to construct nonnegative $f_n \nearrow f$.

Since f_n is continuous on A for each n, we may compute

$$\int_A f_n \, d\mu = \int_0^1 f_n(x) \, dx = 1 + \log n$$

using antiderivatives and Riemann's integral. Then Th.28.2(MCT) implies

$$\int_A f \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu = \lim_{n\to\infty} (1 + \log n) = +\infty,$$

so $f \notin \mathcal{L}(A)$.

(b) Put $\{g_n \stackrel{\text{def}}{=} \min(g, \sqrt{n}) : n = 1, 2, \ldots\}$. As in part (a), g_n is continuous on A for each n and we may compute

$$\int_A g_n \, d\mu = \int_0^1 g_n(x) \, dx = \int_0^{1/n} \sqrt{n} \, dx + \int_{1/n}^1 \frac{1}{\sqrt{x}} \, dx = 2 - \frac{1}{\sqrt{n}}$$

using antiderivatives and Riemann's integral. Then Th.28.2(MCT) implies

$$\int_A g \, d\mu = \lim_{n\to\infty} \int_A g_n \, d\mu = \lim_{n\to\infty} (2 - \frac{1}{\sqrt{n}}) = 2.$$

□

7. (Ex.31.11,p.89) Show that strict inequality may hold in the conclusion of Fatou's Lemma (28.7, p.81)

Solution: There are many examples but an easy one is $A = [0, 1] \subset \mathbf{R}$, with

$$f_n \stackrel{\text{def}}{=} n \, \chi_{(\frac{1}{n}, \frac{2}{n})} \to f \equiv 0.$$

Then $\int_A f_n \, d\mu = 1$ for all $n = 1, 2, \ldots$, so

$$\int_A f \, d\mu = 0 < 1 = \liminf_{n\to\infty} \int_A f_n \, d\mu.$$

□

8. (Ex.31.20,p.89) Let $A \subset \mathbf{R}$ be bounded and measurable. Suppose $f, h \in \mathcal{L}(A)$ with $\int_A f \, d\mu = \int_A h \, d\mu$. Show that if $f \leq g \leq h$ on A, then $g \in \mathcal{L}(A)$ and $\int_A g \, d\mu = \int_A f \, d\mu$.

Solution: Idea: show that $g = f$ a.e. on A.

Let $B \stackrel{\text{def}}{=} \{x \in A : f(x) < h(x)\}$ and write

$$B = \bigcup_{n=1}^{\infty} B_n, \qquad B_n \stackrel{\text{def}}{=} \{x \in A : h(x) - f(x) \geq \frac{1}{n}\},$$

noting that B_n is measurable for every n. Also, $A \setminus B = \{x \in A : f(x) = h(x)\}$.

Since $h - f \geq 0$ and $\int_A f = \int_A h$, we have

$$0 = \int_A h \, d\mu - \int_A f \, d\mu = \int_A (h - f) \, d\mu \geq \int_{B_n} (h - f) \, d\mu \geq \frac{1}{n} \mu(B_n) \geq 0$$

from which it follows that $\mu(B_n) = 0$. But then $\mu(B) = 0$ by countable additivity, so $f = h$ a.e. Conclude from $f \leq g \leq h$ that $f = g$ a.e., and thus that $g \in \mathcal{L}(A)$ by Cor.26.5(2). □

9. (Ex.31.21,p.90) Give an example to show that Egoroff's Theorem (30.1, p.85) cannot be improved to yield $\mu(A_\epsilon) = 0$.

Solution: Idea: If $f_n \to f$ uniformly on $A_0 \subset A$, then $\sup\{|f_n(x) - f(x)| : x \in A_0\} \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in A_0\} = 0.$$

To show that f_n fails to converge uniformly on a particular A_0, it therefore suffices to find a sequence $\{x_n\} \subset A_0$ such that $|f_n(x_n) - f(x_n)|$ is bounded away from 0 for all n.

Let $A = (0,1) \subset \mathbf{R}$, and let $\{f_n : n = 1, 2, \ldots\}$ be the monotonic sequence of nonnegative functions defined by

$$f_n(x) = \begin{cases} 1, & 0 < x < \frac{1}{n}, \\ 0, & \frac{1}{n} \leq x < 1. \end{cases}$$

Then $f_n(x) \searrow f(x) \stackrel{\text{def}}{=} 0$ for all $x \in A$. We claim that any $A_0 \subset A$ with $\mu(A_0) = \mu(A)$ forces the existence of $\{x_n\} \subset A_0$ with $f_n(x_n) = 1$, all $n = 1, 2, \ldots$

Namely, suppose that $A_0 \subset A$ is measurable with $\mu(A_0) = \mu(A) = 1$. Then

$$(\forall n = 1, 2, \ldots)(\exists x_n \in A_0)\ 0 < x_n < \frac{1}{n},$$

since otherwise there is a subinterval $(0, 1/n) \subset A \setminus A_0$ for some positive integer n, which implies the contradiction $\mu(A_0) \leq 1 - \frac{1}{n} < 1$.

But then since $S \stackrel{\text{def}}{=} \{x_n : n = 1, 2, \ldots\} \subset A_0$,

$$\lim_{n\to\infty} \sup\{f_n(x) : x \in A_0\} \geq \lim_{n\to\infty} \sup\{f_n(x) : x \in S\} \geq \lim_{n\to\infty} 1 = 1,$$

conclude that $\{f_n\}$ does not converge uniformly on A_0. □

10. (Ex.31.25,p.90) Let $f = \chi_{\mathbf{Q}}$ on $[0,1]$, Given $\epsilon > 0$, find C_ϵ as in Lusin's Theorem (30.3, p.87).

Solution: Idea: exclude points of discontinuity within tiny open intervals.

Let $\mathbf{Q} \cap [0,1] = \{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers in $[0,1]$.

Now fix $\delta > 0$. For each $n = 1, 2, \ldots$, the interval

$$I_n \stackrel{\text{def}}{=} (q_n - \delta/2^n, q_n + \delta/2^n) \cap [0,1]$$

is relatively open, so

$$F_n \stackrel{\text{def}}{=} [0,1] \setminus \bigcup_{j=1}^{n} I_j, \qquad n = 1, 2, \ldots$$

is a decreasing nested sequence of closed and bounded sets:

$$[0,1] \supset F_1 \supset F_2 \subset F_3 \subset \cdots$$

satisfying (by countable subadditivity)

$$\mu(F_n) \geq 1 - \sum_{j=1}^{n} \mu(I_j) \geq 1 - 2\sum_{j=1}^{n} \frac{\delta}{2^j} \geq 1 - 2\delta.$$

In particular, for any $\delta < 1/2$ the sets F_n are nonempty for all n, so their intersection is a nonempty closed set, whose measure may be estimated using Cor.10.10(p.32):

$$F_\delta \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} F_n \qquad \Rightarrow \qquad \mu(F_\delta) = \lim_{n\to\infty} \mu(F_n) \geq 1 - 2\delta.$$

(The limit exists because $\{\mu(F_n) : n = 1, 2, \ldots\}$ is decreasing and bounded below.) But then

$$\mu([0,1] \setminus F_\delta) = \mu([0,1]) - \mu(F_\delta) \leq 2\delta.$$

Since $\delta > 0$ was arbitrary, for any given $\epsilon > 0$ we may take $0 < \delta < \min\{1/2, \epsilon/2\}$ and put $C_\epsilon \overset{\text{def}}{=} F_\delta$ to get the Lusin set. Then $\chi_{\mathbf{Q}} \equiv 0$ on C_ϵ and is evidently continuous. □