Ma 4102: Introduction to Lebesgue Integration
Solutions to Homework Assignment 4

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Read Chapter 7 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Note: Many of the exercises in section 34 of Chapter 7, in addition to those assigned, are worthy of your efforts.

1. (Ex.34.3,p.108) Let $M \stackrel{\text{def}}{=} \{f : [0,1] \to \mathbf{R} | f$ attains both a minimum and a maximum value.$\}$ Show that M is not a vector space, namely it is not preserved by addition and scalar multiplication.

   Solution: Many counterexamples exist, all sharing discontinuities as a key property. For example, let $f(x) = x$, and let

   $$g(x) = \begin{cases} 1/2, & x = 0 \\ -1/2, & x = 1 \\ 0, & 0 < x < 1. \end{cases}$$

   Then $f, g \in M$, attaining their extrema at 0 and 1, but $f + g$ attains neither its maximum nor its minimum in $[0,1]$.

   (The continuous functions in M do form a vector space.) □

2. (Ex.34.5 and 34.18,p.109)

   (a) For any norm $\|\cdot\|$, prove that $|\|x\| - \|y\|| \leq \|x - y\|$

   (b) For the derived norm $\|x\|$, prove that $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

   (c) Find a norm for which (b) does not hold.

   Solution: (a) Since $\|u + v\| \leq \|u\| + \|v\|$ and $\|-u\| = \|u\|$ for any norm, put $u = x - y$ and $v = y$ to get

   $$\|x\| \leq \|x - y\| + \|y\|, \quad \Rightarrow \|x\| - \|y\| \leq \|x - y\|.$$

Similarly, put $u = y - x$ and $v = x$ to get

$$\|y\| \leq \|y - x\| + \|x\| = \|x - y\| + \|x\|, \quad \Rightarrow \|y\| - \|x\| \leq \|x - y\|.$$

Conclude that $|\|x\| - \|y\|| \leq \|x - y\|$.

(b) Expand, using the linearity and symmetry of the inner product from which $\|\cdot\|$ is derived:

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2x \cdot y + \|y\|^2, \\ \|x - y\|^2 &= \|x\|^2 - 2x \cdot y + \|y\|^2. \end{aligned}$$

Adding both sides yields the result.

(c) There are many examples but the supremum norm on $\mathbf{R}^2$ gives an easy one:

$$x = (1, 0), y = (0, 1) \quad \Rightarrow \|x\| = \|y\| = \|x + y\| = \|x - y\| = 1,$$

so $\|x + y\|^2 + \|x - y\|^2 = 2$ while $2(\|x\|^2 + \|y\|^2) = 4$. □

3. (Ex.34.8,p.109) Prove that if $f_n \in C([0, 1])$ for $n = 1, 2, \ldots,$ and $f_n \to f$ in the supremum norm (Example 32.4(2),p.95), then $f_n \to f$ in the $\mathcal{L}^1$ norm (Example 32.4(3),p.95).

   Solution: First note that f is continuous since it is the uniform limit of a sequence of continuous functions, so $|f_n - f|$ is continuous for every n.

   Now let $\epsilon > 0$ be given and find N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$. Then by monotonicity of the Riemann integral,

$$\|f_n - f\|_1 = \int_0^1 |f_n(x) - f(x)|\, dx < \int_0^1 \epsilon\, dx = \epsilon$$

   Conclude that $\lim_{n \to \infty} \|f_n - f\|_1 = 0$, so $f_n \to f$ in $\mathcal{L}^1$ norm. □

4. (Ex.34.9,p.109) Prove that any convergent sequence in a normed linear space is a Cauchy sequence.

   Solution: Suppose $x_n \to L$ as $n \to \infty$. Let $\epsilon > 0$ be given. Then there is some $N < \infty$ such that

$$n \geq N \quad \Rightarrow \|x_n - L\| < \epsilon/2.$$

   But then by the triangle inequality,

$$n, m \geq N \quad \Rightarrow \|x_n - x_m\| = \|x_n - L + (L - x_m)\| \leq \|x_n - L\| + \|x_n - L\| < \epsilon,$$

   and so $\{x_n\}$ is a Cauchy sequence. □

5. (Ex.34.21,p.110, corrected)

   (a) Prove that an orthonormal set in an inner product space is linearly independent.

   (b) Prove that if $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V, then any $v \in V$ can be expressed as

$$v = \sum_{i=1}^{n} (v \cdot v_i) v_i$$

   Solution: (a) Suppose that $\{v_1, \ldots, v_n\}$ is an orthonormal set in an inner product space, and that $\{c_1, \ldots, c_n\} \subset \mathbf{R}$ with $c_1 v_1 + \cdots + c_n v_n = 0$. For $j \in \{1, \ldots, n\}$, compute

$$0 = v_j \cdot 0 = v_j \cdot \left(\sum_{i=1}^{n} c_i v_i \right) = \sum_{i=1}^{n} c_i (v_j \cdot v_i) = c_j (v_j \cdot v_j) = c_j.$$

   Thus $c_j = 0$ for all j. Conclude that $\{v_1, \ldots, v_n\}$ is a linearly independent set.

   (b) Since $\{v_1, \ldots, v_n\}$ is a basis, any $v \in V$ may be written as

$$v = \sum_{i=1}^{n} c_i v_i$$

   for some unique $(c_1, \ldots, c_n) \in \mathbf{R}^n$. But then

$$v \cdot v_j = \left(\sum_{i=1}^{n} c_i v_i \right) \cdot v_j = \sum_{i=1}^{n} c_i (v_i \cdot v_j) = c_j (v_j \cdot v_j) = c_j,$$

   since $v_j \cdot v_j = 1$ while $v_i \cdot v_j = 0$ for $i \neq j$ in the orthonormal set $\{v_1, \ldots, v_n\}$. Conclude that $c_i = v \cdot v_i$ for each $i = 1, \ldots, n$. □

6. (Ex.34.29 and 34.30,p.111)

   (a) Prove that if $f_n, f \in \mathcal{L}^2(A)$ and $f_n \to f$ uniformly on A, then $f_n \to f$ in $\mathcal{L}^2$.

   (b) Find a counterexample to show that pointwise convergence does not imply $\mathcal{L}^2$ convergence.

   Solution: (a) Recall that we assume that A is bounded and measurable in the definition of $\mathcal{L}^2(A)$, so $\mu(A) < \infty$.

   Let $\epsilon > 0$ be given. By uniform convergence, there exists N such that

$$n \geq N \quad \Rightarrow (\forall x \in A) |f_n(x) - f(x)| < \sqrt{\frac{\epsilon}{\mu(A)}}.$$

From the hypotheses we know that $|f_n - f| \in \mathcal{L}^2(A)$ for every n, so we may compute

$$\|f_n - f\|_2^2 = \int_A |f_n - f|^2 \, d\mu < \frac{\epsilon}{\mu(A)} \, \mu(A) = \epsilon.$$

Since ϵ was arbitrary, we may conclude that $\lim_{n\to\infty} \|f_n - f\|_2 = 0$.

NOTE: the result fails for unbounded A with $\mu(A) = \infty$. One counterexample with $A = [0, \infty)$ is

$$f_n(x) = \begin{cases} 1/\sqrt{n}, & 0 \le x \le n, \\ 0, & \text{otherwise}, \end{cases}$$

which converges uniformly to 0 on A but satisfies $\|f_n\|_2 = 1$ for every n.

(b) There are many pointwise convergent sequences that evade dominated convergence such as:

$$f_n(x) = \begin{cases} \sqrt{n}, & \frac{1}{n} < x \le \frac{2}{n}, \\ 0, & \text{otherwise}, \end{cases}$$

for $x \in [0, 2]$. Then $f_n \to 0$ pointwise as $n \to \infty$, but $\|f_n - 0\|_2 = \|f_n\|_2 = 1$ for every n, so $\lim_{n\to\infty} \|f_n - 0\|_2 = 1 \neq 0$. □

7. (Ex.34.32,p.111) Let $f_n, f, g \in \mathcal{L}^2(A)$, and suppose $f_n \to f$ in $\mathcal{L}^2$. Prove that

$$\lim_{n\to\infty} \int_A f_n g \, d\mu = \int_A f g \, d\mu.$$

Solution: Apply Cauchy's inequality using the inner product $f \cdot g = \int_A f g \, d\mu$ of $\mathcal{L}^2(A)$ and its derived norm $\|f\|_2 = \sqrt{f \cdot f}$:

$$\left| \int_A f_n g \, d\mu - \int_A f g \, d\mu \right| = |(f_n - f) \cdot g| \le \|f_n - f\|_2 \, \|g\|_2 \to 0, \quad \text{as } n \to \infty,$$

since $\lim_{n\to\infty} \|f_n - f\|_2 = 0$ by hypothesis. □

8. (Ex.34.36,p.111) Show that the unit sphere $\{f \in \mathcal{L}^2([0,1]) : \|f\| = 1\}$ is not compact. [Hint: find a sequence $\{g_n : n = 1, 2, \ldots\}$ with $\|g\| = 1$ but with $\|g_n - g_m\| \ge 1$ for all $n \neq m$.]

Solution: Following the hint, let

$$g_n(x) = \begin{cases} 2^{-n/2}, & 2^{-n} < x < 2 \times 2^{-n} \\ 0, & \text{otherwise}, \end{cases}$$

for $n = 1, 2, \ldots$. Then $\|g_n\| = 1$ for every n, but $\|g_n - g_m\| = \sqrt{2}$ whenever $n \neq m$. Hence the infinite set $\{g_n : n = 1, 2, \ldots\}$ has no accumulation point in the unit sphere, which it would if the unit sphere were compact. □

9. (Ex.34.33,p.111) Prove that if $f_n \in \mathcal{L}^2(A)$ for all n, and $f_n \to f$ in $\mathcal{L}^2$, then $f \in \mathcal{L}^2(A)$.

Solution: In other words, prove that $\mathcal{L}^2(A)$ is complete. As always, assume that A is bounded and measurable.

First use Ex.4 above to conclude that $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^2(A)$. Then apply Th.33.11, p.105, to conclude that $\{f_n\}$ converges pointwise almost everywhere to a function $\tilde{f} \in \mathcal{L}^2(A)$, and that $\lim_{n\to\infty} \|f_n - \tilde{f}\|_2 = 0$.

Finally, since $f_n \to f$ in $\mathcal{L}^2(A)$, use the triangle inequality to estimate

$$0 \le \|f - \tilde{f}\|_2 \le \|f - f_n\|_2 + \|f_n - \tilde{f}\|_2 \to 0, \quad \text{as } n \to \infty,$$

so $\|f - \tilde{f}\|_2 = 0$, so $f = \tilde{f}$ a.e., so $f \in \mathcal{L}^2(A)$. □

10. (Ex.34.41,p.112) Suppose that $A \subset \mathbf{R}$ is bounded and Lebesgue measurable, and f is a measurable function defined on A.

(a) Show that $\|f\|_1 \le \|f\|_2 \sqrt{\mu(A)}$.

(b) Find a sequence $\{f_n\} \subset \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$ which converges to 0 in $\mathcal{L}^1$ but does not converge in $\mathcal{L}^2$. □

Solution: (a) Since $\mu(A) < \infty$, the constant function 1 belongs to $\mathcal{L}^2(A)$ with $\|1\|_2 = \sqrt{\mu(A)}$. Use this with Cauchy's inequality to compute

$$\|f\|_1 = \int_A |f|\, d\mu = |f| \cdot 1 \le \|f\|_2 \|1\|_2 = \|f\|_2 \sqrt{\mu(A)}.$$

Note that this implies $\mathcal{L}^2(A) \subset \mathcal{L}^1(A)$ for bounded measurable $A \subset \mathbf{R}$.

(b) One source of counterexamples are sequences with nondecreasing $\mathcal{L}^2$ norm, such as the one from Ex.6(b) above:

$$f_n(x) = \begin{cases} \sqrt{n}, & \frac{1}{n} < x \le \frac{2}{n}, \\ 0, & \text{otherwise,} \end{cases}$$

for $n = 1, 2, \ldots$ and $x \in [0, 2]$. Then

$$\|f_n - 0\|_1 = \int_{[0,2]} |f_n|\, d\mu = \int_0^2 f_n(x)\, dx = \int_{1/n}^{2/n} \sqrt{n}\, dx = 1/\sqrt{n},$$

so $f_n \to 0$ in $\mathcal{L}^1([0,1])$ as $n \to \infty$, but $\|f_n - 0\|_2 = \|f_n\|_2 = 1$ for every n, so $\lim_{n\to\infty} \|f_n - 0\|_2 = 1 \ne 0$. □