Ma 4102: Introduction to Lebesgue Integration
Solutions to Homework Assignment 5

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Read Chapter 8 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Note: Many of the exercises in section 38 of Chapter 8, in addition to those assigned, are worthy of your efforts.

1. (Ex.38.1,p.127) A function $f \in \mathcal{L}^2([-\pi,\pi])$ is said to be *even* iff $f(-x) = f(x)$ for all $x \in [-\pi,\pi]$. It is said to be *odd* iff $f(-x) = -f(x)$ for all $x \in [-\pi,\pi]$. Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for f.

(a) Prove that if f is even, then $b_n = 0$ for all $n = 1, 2, \ldots$

(b) Prove that if f is odd, then $a_n = 0$ for all $n = 0, 1, 2, \ldots$

Solution: First show that if $g \in \mathcal{L}^2([-\pi,\pi]) \subset \mathcal{L}([-\pi,\pi])$ is odd, then $\int_{[-\pi,\pi]} g\, d\mu = 0$. This may be shown in three steps.

First, it is true for odd simple functions g by direct calculation using Prop.22.1, p.61. Furthermore, an arbitrary simple function g defined on $(0,\pi]$ may be extended to an odd simple function on $[-\pi,\pi]$ by defining $g(0) = 0$ and $g(-x) = -g(x)$ for all $0 < x \leq \pi$.

Second, given odd $f \in \mathcal{L}^2([-\pi,\pi]) \subset \mathcal{L}([-\pi,\pi])$, note that $f \in \mathcal{L}^2((0,\pi]) \subset \mathcal{L}((0,\pi])$, so by Th.23.1, p.62, for every $\epsilon > 0$ there exists a simple function g satisfying

$$\int_{(0,\pi]} |f-g|\, d\mu < \epsilon/2.$$

Extend this g to an odd simple functions on $[-\pi,\pi]$ as above and note that

$$\int_{[-\pi,\pi]} |f-g|\, d\mu < \epsilon.$$

Finally, apply the triangle inequality:

$$\left| \int_{[-\pi,\pi]} f\, d\mu - 0 \right| = \left| \int_{[-\pi,\pi]} f\, d\mu - \int_{[-\pi,\pi]} g\, d\mu \right| \leq \int_{[-\pi,\pi]} |f-g|\, d\mu < \epsilon,$$

and since $\epsilon > 0$ was arbitrary, conclude that $\int_{[-\pi,\pi]} f\, d\mu = 0$.

(a) If f is even, then $f(x)\sin nx$ is odd since $\sin(-\theta) = -\sin\theta$ for every θ. But then

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0,$$

for all $n = 1, 2, \ldots$.

(b) If f is odd, then $f(x)\cos nx$ is odd since $\cos(-\theta) = \cos\theta$ for every θ. But then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0,$$

for all $n = 0, 1, 2, \ldots$. □

2. (Ex.38.9,p.128) Suppose that $\sum_{k=1}^{\infty} a_k$ is an infinite series with $a_k \geq 0$ for every $k = 1, 2, \ldots$. Let σ_n be the arithmetic mean of its first n partial sums as in Def.37.1, p.121.

(a) Prove that $\sigma_n \leq \sigma_{n+1}$ for every $n = 1, 2, \ldots$

(b) Prove that if $\sum_{k=1}^{\infty} a_k = \infty$, then $\lim_{n\to\infty} \sigma_n = \infty$.

[Conclude that a series with non-negative terms converges to a real number if and only if it $(C, 1)$ converges to that number.]

Solution: Let s_n be the nth partial sum of $\sum a_k$.

(a) Since $s_{n+1} - s_k = a_{n+1} + \cdots + a_{k+1} \geq 0$, it follows that $s_{n+1} \geq s_k$ for all $1 \leq k \leq n$. Thus $s_{n+1} \geq \frac{1}{n}(s_1 + \cdots + s_n) = \sigma_n$, and so

$$\sigma_{n+1} = \frac{s_1 + \cdots + s_n + s_{n+1}}{n+1} = \frac{n\sigma_n + s_{n+1}}{n+1} \geq \frac{n\sigma_n + \sigma_n}{n+1} \geq \frac{(n+1)\sigma_n}{n+1} = \sigma_n.$$

(b) The nonnegative series diverging to infinity means that $0 \leq s_n \nearrow \infty$ as $n \to \infty$.

Let M be given. There exists a number N such that $n > N \implies s_n > M + 1$. But then

$$\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n) = \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n} \geq \frac{n-N}{n}(M+1) > M,$$

for all sufficiently large $n > N$. This is the defnition of $\sigma_n \to \infty$ as $n \to \infty$. □

3. (Ex.38.14,p.129) If $f \in \mathcal{L}([-\pi, \pi])$ is extended periodically to $\mathbf{R}$, show that for any $x \in \mathbf{R}$,

$$\int_{[x-\pi,x+\pi]} f\, d\mu = \int_{[-\pi,\pi]} f\, d\mu$$

[HINT: draw a picture.]

Solution: First suppose that $0 \leq x < 2\pi$. Put $A = [-\pi, x-\pi]$, $B = [x-\pi, \pi]$ and $C = A + 2\pi = [\pi, \pi + x]$ so that

- $[-\pi, \pi] = A \cup B$ is a disjoint decomposition except for a single point in $A \cap B$, while
- $[x - \pi, x + \pi] = B \cup C$ is a disjoint decomposition except for a single point in $B \cap C$.

Now use the result of Exercise 26.34, p.76, which states that if $g \in \mathcal{L}(E)$ and $F = E+c = \{t+c : t \in E\}$, then the function $h(t) \overset{\text{def}}{=} g(t-c)$ belongs to $\mathcal{L}(F)$ and

$$\int_F h \, d\mu = \int_E g \, d\mu.$$

Apply this to $E = A$, $F = A + 2\pi = C$, $g(t) = f(t)$, and $h(t) = f(t - 2\pi) = f(t)$ (by 2π-periodicity) to get

$$\int_A f \, d\mu = \int_C f \, d\mu.$$

The result follows from this and Theorem 25.3, p.70:

$$\int_{A\cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu = \int_C f \, d\mu + \int_B f \, d\mu = \int_{B\cup C} f \, d\mu.$$

Finally, for any $x \in \mathbf{R}$ there is some integer n such that $0 \le x + 2n\pi < 2\pi$. The result holds for this x as well since it holds for $x + 2n\pi$ and $f(t + 2n\pi) = f(t)$ for all $t \in \mathbf{R}$. □

4. (Ex.38.16,p.129) Prove the following analogue to Lem.37.7, p.125:

   (a) $\int_0^\pi D_n(t) \, dt = \pi/2$, and

   (b) for $0 < \delta \le |t| \le \pi$, we have the inequality $|D_n(t)| \le \dfrac{1}{2|\sin(\delta/2)|}$.

   Solution:

   (a) Let $f(x) = 1$ at all $x \in [-\pi, \pi]$. Then $s_n = 1$ for all $n = 0, 1, \ldots,$ and $f(x+t) + f(x-t) = 2$ for all x, t, so by Lemma 37.5 on p.123,

   $$1 = \frac{2}{\pi} \int_{-\pi}^{\pi} D_n(t) \, dt, \qquad n = 0, 1, 2, \ldots,$$

   giving the desired result.

   (b) Since $\sin(t/2)$ is strictly increasing on $0 \le t \le \pi$, and $|\sin x| = \sin |x|$ for all x, we have the denominator lower bound $|\sin(t/2)| = \sin(|t/2|) \ge \sin(\delta/2) > 0$ for $0 < \delta \le |t| \le \pi$. Likewise, we have the numerator upper bound $|\sin(n + \frac{1}{2})t| \le 1$ for all $t \in \mathbf{R}$. Combining these gives

   $$|D_n(t)| = \left| \frac{\sin(n + \frac{1}{2})t}{2\sin(t/2)} \right| \le \frac{1}{2\sin(\delta/2)},$$

   for $0 < \delta \le |t| \le \pi$. □

5. (Ex.38.18,p.129) Prove the following:

   (a) If f is continuous on $[-\pi, \pi]$ and its Fourier coefficients are all 0, then $f \equiv 0$.

   (b) If f and g are continuous on $[-\pi, \pi]$ and the Fourier coefficients of f and g are identical, then $f(x) = g(x)$ for all x.

   (c) If f and g belong to $\mathcal{L}^2([-\pi, \pi])$ and have identical Fourier coefficients, then $f(x) = g(x)$ a.e.$x \in [-\pi, \pi]$.

Solution: (a) All Fourier coefficients are zero, so $s_n(x) = 0$ and thus $\sigma_n(x) = 0$ for all $x \in [-\pi, \pi]$ and all $n = 0, 1, 2, \ldots$.

By Theorem 37.8 on page 125, $\sigma_n \to f$ uniformly on $[-\pi, \pi]$. In particular, $\sigma_n(x) \to f(x)$ at each $x \in [-\pi, \pi]$ as $n \to \infty$, so

$$|f(x)| = |f(x) - 0| = \lim_{n\to\infty} |f(x) - 0| = \lim_{n\to\infty} |f(x) - \sigma_n(x)| = 0,$$

and thus $f(x) = 0$.

(b) Put $h = f - g$. Then h is continuous, and by linearity of the integral each Fourier coefficient of h is zero. Then $h \equiv 0$ by part (a), so $f = g$.

(c) Put $h = f - g$. Then $h \in \mathcal{L}^2([-\pi, \pi]) \subset \mathcal{L}([-\pi, \pi])$. By linearity of the Lebesgue integral each Fourier coefficient of h is zero, so its Fourier series partial sums s_n are identically zero. By Theorem 37.9,

$$\|h\|_2 = \|h - 0\|_2 = \lim_{n\to\infty} \|h - 0\|_2 = \lim_{n\to\infty} \|h - s_n\|_2 = 0,$$

so $h = 0$ a.e., so $f = g$ a.e. □

6. (Ex.38.20,p.129) Prove Parseval's formula: if $f \in \mathcal{L}^2([-\pi, \pi])$, then

$$\frac{1}{\pi} \|f\|_2^2 \; = \; \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

[HINT: compute $\|s_n - f\|_2^2$ and apply Th.37.9.]

Solution: Let $s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ be the nth partial sum of the Fourier series for f. Then

$$\|s_n\|_2^2 = \pi \left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right],$$

evaluated using the orthonormality of the set

$$\{ \frac{1}{\sqrt{2\pi}}; \frac{1}{\sqrt{\pi}} \sin kx, \frac{1}{\sqrt{\pi}} \cos kx, k = 1, 2, \ldots \} \subset C([-\pi, \pi]) \subset \mathcal{L}^2([-\pi, \pi])$$

with respect to the derived $\mathcal{L}^2$ norm $\|\cdot\|_2$ as asserted in Example 32.17(3) on page 99. Thus Parseval's formula follows from the statement

$$\|f\|_2^2 = \lim_{n\to\infty} \|s_n\|_2^2 = \pi \left[\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \right].$$

To prove that, use Ex.34.5, p.109, from HW 4, applied to $\|\cdot\|_2$: for any $u, v \in \mathcal{L}^2$,

$$\Big| \|u\|_2 - \|v\|_2 \Big| \leq \|u - v\|_2.$$

Following the hint, apply this to $u = s_n$ and $v = f$ to get

$$\Big| \|s_n\|_2^2 - \|f\|_2^2 \Big| = \Big| \|s_n\|_2 + \|f\|_2 \Big| \times \Big| \|s_n\|_2 - \|f\|_2 \Big| \leq \Big| \|s_n\|_2 + \|f\|_2 \Big| \times \|s_n - f\|_2.$$

The first factor is bounded between $\|f\|_2$ and $2\|f\|_2$ by Bessel's inequality, Corollary 36.3 on page 118, while the second factor tends to zero, as $n \to \infty$, by Theorem 37.9. □

7. (Ex.38.22,p.130) Suppose that $f, g \in \mathcal{L}^2([-\pi, \pi])$ have respective Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \qquad \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx).$$

Use Parseval's formula with $\|f+g\|_2^2$ and $\|f-g\|_2^2$ to show that

$$\frac{1}{\pi}(f \cdot g) = \frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k)$$

[This is a major application of the "polarization identity."]

Solution: From the definition of derived norm, using properties of the inner product, compute

$$\|f+g\|_2^2 - \|f-g\|_2^2 = (f+g)\cdot(f+g) - (f-g)\cdot(f-g) = 4(f \cdot g).$$

Alternatively, find the Fourier series of $f+g$ and $f-g$ by linearity and apply Parseval's formula to get

$$\begin{aligned} \frac{1}{\pi}\|f+g\|_2^2 &= \frac{(a_0+\alpha_0)^2}{2} + \sum_{k=1}^{\infty}[(a_k+\alpha_k)^2 + (b_k+\beta_k)^2], \\ \frac{1}{\pi}\|f-g\|_2^2 &= \frac{(a_0-\alpha_0)^2}{2} + \sum_{k=1}^{\infty}[(a_k-\alpha_k)^2 + (b_k-\beta_k)^2], \end{aligned}$$

from which we compute the difference term-by-term:

$$\|f+g\|_2^2 - \|f-g\|_2^2 = 4\pi\left(\frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty}[a_k\alpha_k + b_k\beta_k]\right).$$

Equating these two expressions for $\|f+g\|_2^2 - \|f-g\|_2^2$ gives the result. ☐

8. (Ex.38.27,p.130) If f is 2π-periodic and continuously differentiable on $[-\pi, \pi]$, show that the Fourier transform of f' can be obtained from that of f by differentiating term-by-term.

[HINT: integrate by parts.]

Solution: Computing the Fourier coefficients of f' (call them a_k' and b_k') directly gives

$$\begin{aligned} \pi a_k' &= \int_{-\pi}^{\pi} f'(x)\cos kx\,dx \\ &= f(\pi)\cos(k\pi) - f(-\pi)\cos(-k\pi) - \int_{-\pi}^{\pi} f(x)\frac{d}{dx}[\cos kx]\,dx \\ &= 0 - \int_{-\pi}^{\pi} f(x)[-k\sin kx]\,dx \;=\; \pi k b_k, \qquad \Longrightarrow \boxed{a_k' = kb_k} \end{aligned}$$

using 2π periodicity of $f(x)$ and $\cos kx$ to cancel the boundary term $f(\pi)\cos(k\pi) - f(-\pi)\cos(-k\pi)$. Note that the integrands are continuous so we may use Riemann integrals and integration by parts.

Likewise,

$$\begin{aligned}\pi b_k' &= \int_{-\pi}^{\pi} f'(x)\sin kx\,dx \\ &= f(\pi)\sin(k\pi) - f(-\pi)\sin(-k\pi) - \int_{-\pi}^{\pi} f(x)\frac{d}{dx}[\sin kx]\,dx \\ &= 0 - \int_{-\pi}^{\pi} f(x)[k\cos kx]\,dx \;=\; -\pi k a_k, \qquad \Longrightarrow \boxed{b_k' = -ka_k}\end{aligned}$$

Alternatively, term-by-term differentiation of the Fourier series gives

$$f'(x) = \frac{d}{dx}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)\right) = 0 + \sum_{k=1}^{\infty}(kb_k\cos kx - ka_k\sin kx).$$

These agree with the coefficients obtained through integration. ☐

9. (Ex.38.29,p.131) Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are absolutely convergent series. Show that

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)$$

converges absolutely and uniformly on $[-\pi,\pi]$ to a continuous function.

[HINT: Use the Weierstrass M test.]

Solution: Following the hint: the Weierstrass theorem states that if a sequence of functions $\{g_n\}$ defined on a set A satisfies

- $(\exists M_n < \infty)|g_n(x)| \le M_n$ for all $x \in A$, and
- $\sum_n M_n < \infty$ (so the M series converges, in fact absolutely since it is nonnegative),

then $\sum_n g_n$ converges absolutely and uniformly on A.

Apply Weierstrass with the observation that $|a_k\cos kx| \le |a_k|$ and $|b_k\sin kx| \le |b_k|$, all k, and $\sum_k |a_k|$ and $\sum_k |b_k|$ both converge, so the Fourier series converges absolutely and uniformly to f.

Finally, note that since $\sin kx$ and $\cos kx$ are continuous functions for every k, every partial sum $s_n(x)$ is continuous. Therefore, by the uniform limit theorem, $\lim_{n\to\infty} s_n$ is continuous. ☐

10. (Ex.38.30,p.131) Show that if f is continuous on $[-\pi,\pi]$ and the Fourier transform of f converges at x, then it converges to $f(x)$ at x.

[HINT: apply Th.37.8, p.125.]

Solution: Let $\{s_n\}$ be the partial sums of the Fourier series for f, and let $\{\sigma_n\}$ be the corresponding sequence of Cesàro means of these partial sums. By Theorem 37.8, $\sigma_n \to f$ uniformly on $[-\pi,\pi]$, so in particular $\sigma_n(x) \to f(x)$ as $n \to \infty$.

Now suppose that $s_n(x)$ converges to some limit as $n \to \infty$. By Example 37.3, p.121, the Cesàro means converge to that same limit, which for them is $f(x)$. Conclude that $s_n(x) \to f(x)$ as $n \to \infty$, as claimed. ☐