

Ma 4102: Introduction to Lebesgue Integration

Solutions to Homework Assignment 6

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Read Chapter 9 of our textbook.

Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Note: Many of the exercises in section 42 of Chapter 9, in addition to those assigned, are worthy of your efforts.

1. (Ex.42.1,p.144) Show that if the Fourier series of $f \in \mathcal{L}^2([-\pi, \pi])$ converges pointwise a.e., then it converges pointwise a.e. to f .

[HINT: Show, as in the proof of Th.33.11, p.106, that a subsequence of the Fourier partial sums $\{s_n\}$ converges pointwise a.e. to f .]

Solution: Let $\{s_n\}$ be the sequence of partial sums of the Fourier series for $f \in \mathcal{L}^2([-\pi, \pi])$. By Theorem 37.9, p.126, $s_n \rightarrow f$ in \mathcal{L}^2 , namely

$$\lim_{n \rightarrow \infty} \|s_n - f\|_2 = 0.$$

Following the hint, let $\{s_{n_k} : k = 1, 2, \dots\}$ be a subsequence that converges pointwise a.e. to f . By hypothesis, the full sequence $\{s_n\}$ converges a.e., say to some function g , so the subsequence converges a.e. to g , so $f = g$ a.e., so s_n converges to f pointwise a.e. \square

2. (Ex.42.2,p.144) Consider the Fourier series of Example 35.3 on page 116, namely for the function $\chi_{[0,\pi]}$.
(a) Use it and an appropriate theorem to prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

[HINT: put $x = \pi/2$.]

(b) Let $x = \pi/4$ and obtain another series representation for $\pi/4$.

(c) Which series, (a) or (b), needs fewer terms n to reach the accuracy $|s_n - \pi/4| < 10^{-7}$?

Solution: (a) From p.116, the Fourier coefficients are $a_0 = 1$, $a_k = 0$ if $k > 0$, $b_k = 0$ if k is even, and $b_k = 2/(\pi k)$ if $k \geq 1$ is odd, namely $k = 2m - 1$ for $m = 1, 2, \dots$.

The simple function $\chi_{[0,\pi]}$ belongs to $\mathcal{L}^2([-\pi, \pi])$ and is differentiable at $\pi/2$ so by Theorem 40.2, p.139, its Fourier series converges at $\pi/2$ to the value

$$1 = \chi_{[0,\pi]}(\pi/2) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin\left(m - \frac{1}{2}\right)\pi = \frac{1}{2} + \frac{2}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right).$$

Subtracting $1/2$ from both sides and multiplying by $\pi/2$ yields the result.

(b) $\pi/4$ is another point of differentiability, hence the Fourier series converges there, too, giving

$$1 = \chi_{[0,\pi]}(\pi/4) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin \frac{2m-1}{4} \pi = \frac{1}{2} + \frac{2}{\pi\sqrt{2}} \left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots \right),$$

where two terms are added then two are subtracted. Subtracting $1/2$ from both sides and multiplying by $\pi/2$ yields

$$\frac{\pi}{4} = \frac{1}{\sqrt{2}} \left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots \right)$$

(c) Both series converge at approximately the same rate. A numerical experiment shows that after $m = 2500002$ terms the first series gives the approximation

$$0.785398263397396 \approx \frac{\pi}{4} = 0.785398163397448 \dots,$$

while the second series takes $mm = 3535535$ terms, about 41% more, to get the approximation

$$0.785398263397436 \approx \frac{\pi}{4} = 0.785398163397448 \dots,$$

with the same error. □

3. (Ex.42.4,p.144) Suppose that $f : [-\pi, \pi] \rightarrow \mathbf{R}$ is 2π -periodic, continuous on $[-\pi, \pi]$, and differentiable at $x_0 \in [-\pi, \pi]$. For $t \in (-\pi, \pi)$, $t \neq 0$, define

$$h(t) \stackrel{\text{def}}{=} \frac{f(x_0+t) + f(x_0-t) - 2f(x_0)}{2\sin(t/2)}.$$

Show that there is a number $y_0 \in \mathbf{R}$ such that by defining $h(0) \stackrel{\text{def}}{=} y_0$, the resulting function is continuous on $(-\pi, \pi)$.

Solution: Try computing the limit as $t \rightarrow 0$, using the hypothesis that f is differentiable at x_0 and noting that $\sin(t/2)/(t/2) \rightarrow 1$ as $t \rightarrow 0$:

$$\begin{aligned} \lim_{t \rightarrow 0} h(t) &= \lim_{t \rightarrow 0} \frac{f(x_0+t) - f(x_0) + f(x_0-t) - f(x_0)}{2\sin(t/2)} \\ &= \lim_{t \rightarrow 0} \left[\frac{f(x_0+t) - f(x_0)}{t} - \frac{f(x_0-t) - f(x_0)}{-t} \right] \lim_{t \rightarrow 0} \left[\frac{t/2}{\sin(t/2)} \right] \\ &= f'(x_0) - f'(x_0) = 0. \end{aligned}$$

Since it exists, we may set $y_0 \stackrel{\text{def}}{=} h(0) = \lim_{t \rightarrow 0} h(t) = 0$ to get continuity at $t = 0$. But h is a continuous function at all other $t \in (-\pi, \pi)$. □

4. (Ex.42.6,p.144) Show that if $f \in \mathcal{L}^2([-\pi, \pi])$, then

$$\lim_{n \rightarrow \infty} \int_0^\pi f(t) \sin\left(n + \frac{1}{2}\right)t dt = 0.$$

[HINT: Th.36.4, p.118, and angle addition formulas.]

Solution: Following the hint, write

$$\sin\left(n + \frac{1}{2}\right)t = \sin nt \cos \frac{1}{2}t + \cos nt \sin \frac{1}{2}t,$$

so that

$$\int_0^\pi f(t) \sin\left(n + \frac{1}{2}\right)t dt = \int_0^\pi f_c(t) \sin nt dt + \int_0^\pi f_s(t) \cos nt dt$$

where $f_c(t) = f(t) \cos(t/2)$ and $f_s(t) = f(t) \sin(t/2)$. Both f_c and f_s belong to $\mathcal{L}^2([-\pi, \pi])$ since sine and cosine are continuous and bounded functions, so their Fourier coefficients tend to 0 as $n \rightarrow \infty$ by Theorem 36.4, so all these integrals tend to zero as $n \rightarrow \infty$. \square

5. (Ex.42.8,p.144) Given $f \in \mathcal{L}([-\pi, \pi])$ and $\epsilon > 0$, show that there is a simple function g on $[-\pi, \pi]$ such that

$$\int_{[-\pi, \pi]} |f - g| d\mu < \epsilon.$$

[HINT: use f^+ and f^- .]

Solution: Following the hint, write $f = f^+ - f^-$ and use Definition 25.2, p.70, to conclude from $f \in \mathcal{L}([-\pi, \pi])$ that

$$0 \leq \int_A f^+ d\mu < \infty, \quad 0 \leq \int_A f^- d\mu < \infty,$$

for bounded measurable domain $A = [\pi, \pi] \subset \mathbf{R}$. Then use Definition 25.1, p.69, to compute

$$\sup\left\{\int_A g d\mu : g \text{ simple, } 0 \leq g \leq f^+\right\} = \int_A f^+ d\mu < \infty,$$

which implies that for every $\epsilon > 0$ there is a simple function $0 \leq g_1 \leq f^+$ such that

$$0 \leq \int_A f^+ d\mu - \int_A g_1 d\mu < \epsilon/2.$$

Likewise, there is a simple function $0 \leq g_2 \leq f^-$ such that

$$0 \leq \int_A f^- d\mu - \int_A g_2 d\mu < \epsilon/2.$$

Put $g = g_1 - g_2$, which is a simple function, and apply the triangle inequality to estimate

$$|f - g| = |f^+ - g_1 - f^- + g_2| \leq |f^+ - g_1| + |f^- - g_2|$$

and thus

$$\int_A |f - g| d\mu \leq \int_A |f^+ - g_1| d\mu + \int_A |f^- - g_2| d\mu < \epsilon,$$

giving the result. \square

6. (Ex.42.9,p.144) Prove Theorem 41.7.

Solution: Statement: Suppose that $f \in \mathcal{L}[-\pi, \pi]$, and $f'_R(x)$ and $f'_L(x)$ both exist at the point x . Then the Fourier series for f converges to $m(x) \stackrel{\text{def}}{=} (f(x+) + f(x-))/2$ at x .

Lemma 37.5 applies, as stated in the footnote on p.123, so write

$$\begin{aligned} s_n(x) - m(x) &= \frac{1}{\pi} \int_0^\pi [f(x+t) + f(x-t) - 2m(x)] D_n(t) dt \\ &= \frac{2}{\pi} \int_0^\pi \frac{f(x+t) - f(x-t) + f(x-t) - f(x-)}{t} \frac{t}{2 \sin(t/2)} \sin(n + \frac{1}{2})t dt, \end{aligned}$$

using the fact that $\int_0^\pi D_n(t) dt = \pi/2$ from Ex.38.16, p.129 as in the proof of Th.40.2, p.139, and factoring out the denominator $2 \sin(t/2)$ of the Dirichlet kernel.

The first factor in the integral can be made continuous at $t = 0$ by defining it there to be its limit $f'_R(x) - f'_L(x)$. Elsewhere it is a linear combination of functions in $\mathcal{L}[-\pi, \pi]$, so it too belongs to $\mathcal{L}[-\pi, \pi]$.

The second factor $t/(2 \sin t/2)$ is bounded and continuous on $[0, \pi]$ if we use its right limit 1 at $t = 0$, hence its product with the first factor belongs to $\mathcal{L}[-\pi, \pi]$.

Conclude by the Riemann-Lebesgue lemma (41.1, p.141) that the integral tends to zero, and thus $s_n(x) \rightarrow m(x)$, as $n \rightarrow \infty$. \square

7. (Ex.42.11,p.144)

- (a) For $x \in [0, \pi)$, find the Fourier series (namely, the coefficients $\{a_n, b_n\}$) for the function $\chi_{[0,x)}$.
 (b) Use part (a) and appropriate theorems to obtain the identity

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

Solution: (a) From the definitions and the formula for the function $\chi_{[0,x)}(t)$:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^x dt = \frac{x}{\pi}, \\ a_n &= \frac{1}{\pi} \int_0^x \cos nt dt = \frac{\sin nx}{\pi n}, \quad n = 1, 2, \dots \\ b_n &= \frac{1}{\pi} \int_0^x \sin nt dt = \frac{1 - \cos nx}{\pi n}, \quad n = 1, 2, \dots \end{aligned}$$

(b) Use Theorem 41.7 to compute the limit of the Fourier series for $\chi_{[0,x)}(t)$ at $t = 0$:

$$\frac{1}{2} = \frac{\chi_{[0,x)}(0-) + \chi_{[0,x)}(0+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n0 + b_n \sin n0) = \frac{x}{2\pi} + \sum_{n=1}^{\infty} \frac{\sin nx}{n\pi},$$

and then multiplying by π and subtracting $x/2$ from both sides yields the result. \square

8. (Ex.42.12,p.145) Let $\{a_0; a_n, b_n, n = 1, 2, 3, \dots\} \subset \mathbf{R}$ be the coefficients giving a sequence of trigonometric series

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx), \quad n = 0, 1, 2, \dots$$

(a) Show that if $\{s_n(c)\}$ converges absolutely, then $\{s_n(-c)\}$ also converges absolutely.

- (b) Show that if $\{s_n(c_1)\}$ and $\{s_n(c_2)\}$ both converge absolutely, then $\{s_n(c_1+c_2)\}$ converges absolutely.
(c) Show that if $\{s_n(x)\}$ converges absolutely for all x in some nonempty open interval, then it converges absolutely at all $x \in [-\pi, \pi]$.

Solution: (a) This follows immediately from the identities $|\sin \theta| = |\sin(-\theta)|$ and $|\cos \theta| = |\cos(-\theta)|$, valid for all $\theta \in \mathbf{R}$.

(b) Use the angle addition formulas and the triangle inequality:

$$\begin{aligned} 0 \leq |\sin(\theta_1 + \theta_2)| &= |\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2| \leq |\sin \theta_1| |\cos \theta_2| + |\cos \theta_1| |\sin \theta_2|, \\ 0 \leq |\cos(\theta_1 + \theta_2)| &= |\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2| \leq |\cos \theta_1| |\cos \theta_2| + |\sin \theta_1| |\sin \theta_2|. \end{aligned}$$

Since sine and cosine are each bounded by 1, the Fourier series $s_n(c_1 + c_2)$ converges absolutely by the comparison test with $|s_n(c_1)| + |s_n(c_2)|$.

(c) It suffices to show that, given any nonempty open interval $I = (a, b)$, every $x \in [-\pi, \pi]$ may be written as a finite linear combination of elements of I with integer coefficients. Then induction on part (b) above, getting negative integer coefficients by using (a), proves that $s_n(x)$ converges absolutely.

So, fix $p, q \in I$ with $a < p < q < b$, and put $\delta \stackrel{\text{def}}{=} q - p > 0$. By Archimedes' principle, there is some integer N such that $x + N\delta \in I$, so write $r \stackrel{\text{def}}{=} x + N\delta$ to get

$$x = r - N\delta = r - Nq + Np,$$

which is an integer combination of elements of I . □

9. (Extra A) Using Euler's formula $e^{ix} = \cos x + i \sin x$, it is possible to write a trigonometric polynomial in terms of complex exponential functions:

$$\frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) = \sum_{k=-n}^n c_k e^{ikx}.$$

(a) Find the formulas relating (c_k, c_{-k}) and (a_k, b_k) , for $k = 0, 1, 2, \dots$

(b) Show that $\{e^{ikx} : k \in \mathbf{Z}\}$ is an orthogonal set of functions in $\mathcal{L}^2([-\pi, \pi])$ with respect to the Hermitean inner product

$$f \cdot g \stackrel{\text{def}}{=} \int_{[-\pi, \pi]} f \bar{g} d\mu,$$

where \bar{g} is the complex conjugate of g , so for example $\overline{e^{ix}} = e^{-ix}$ for all real x .

Solution: (a) Expand the terms with $k \neq 0$ using Euler's formula:

$$\begin{aligned} \sum_{k=-n}^n c_k e^{ikx} - c_0 &= \sum_{k=1}^n [c_k e^{ikx} + c_{-k} e^{-ikx}] = \sum_{k=1}^n [c_k (\cos kx + i \sin kx) + c_{-k} (\cos kx - i \sin kx)] \\ &= \sum_{k=1}^n [(c_k + c_{-k}) \cos kx + i(c_k - c_{-k}) \sin kx], \end{aligned}$$

from which it follows that $c_0 = a_0/2$, while $a_k = c_k + c_{-k}$ and $b_k = i(c_k - c_{-k})$, all $k > 0$. In the other direction we have $a_0 = 2c_0$, while for $k > 0$ we have $c_k = (a_k - ib_k)/2$ and $c_{-k} = (a_k + ib_k)/2$.

(b) Check orthogonality by evaluating the integrals: for $I = [-\pi, \pi] \subset \mathbf{R}$ and $n, m \in \mathbf{Z}$,

$$\int_I e^{inx} e^{-imx} d\mu = \int_I e^{i(n-m)x} d\mu = \begin{cases} \int_I 1 d\mu = |I| = 2\pi, & n = m, \\ \int_I [\cos(n-m)x + i \sin(n-m)x] d\mu = 0, & n \neq m, \end{cases}$$

since

$$\int_I \cos kx d\mu = \frac{\sin kx}{k} \Big|_{-\pi}^{\pi} = 0, \quad \int_I \sin kx d\mu = \frac{-\cos kx}{k} \Big|_{-\pi}^{\pi} = 0,$$

for integer $k = n - m \neq 0$ due to the 2π -periodicity of sine and cosine. □

10. (Extra B) For integer $N > 0$, let $A_N \stackrel{\text{def}}{=} [-N, N] \subset \mathbf{R}$. Say that $f \in \mathcal{L}(\mathbf{R})$ if

- $f \in \mathcal{L}(A_N)$ for every $N = 1, 2, 3, \dots$, and
- $\|f\| \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \int_{A_N} |f| d\mu$ exists and is finite.

(a) Prove that if $f \in \mathcal{L}(\mathbf{R})$ and g is a bounded measurable function on \mathbf{R} , then $gf \in \mathcal{L}(\mathbf{R})$.

(b) Prove that if $f, g \in \mathcal{L}(\mathbf{R})$ then $f + g \in \mathcal{L}(\mathbf{R})$.

(c) For $f(x) = e^{-\pi x^2}$, compute

$$\int_{\mathbf{R}} f d\mu = \lim_{N \rightarrow \infty} \int_{A_N} f d\mu$$

[HINT: With justification, use results from Calculus.]

Solution: (a) Since f is measurable (as the pointwise limit of measurable functions $f\chi_{A_N}$), and measurability is preserved by products, we know that gf is measurable.

Suppose $|g| \leq M < \infty$ gives an upper bound for g . Then by monotonicity,

$$\lim_{N \rightarrow \infty} \int_{A_N} |gf| d\mu \leq M \lim_{N \rightarrow \infty} \int_{A_N} |f| d\mu = M\|f\| < \infty,$$

(b) Since $f + g \in \mathcal{L}(A_N)$ for all N by results for bounded domains, it remains only to check

$$\lim_{N \rightarrow \infty} \int_{A_N} |f + g| d\mu \leq \int_{A_N} |f| d\mu + \int_{A_N} |g| d\mu < \infty,$$

which follows from the hypothesis $f, g \in \mathcal{L}(\mathbf{R})$ and the triangle inequality on each $\mathcal{L}(A_N)$.

(c) Since $f(x) > 0$ for all $x \in \mathbf{R}$, it suffices to estimate via Taylor expansions

$$0 < f(x) = e^{-\pi x^2} = \frac{1}{e^{\pi x^2}} = \frac{1}{1 + \pi x^2 + \dots} \leq \frac{1}{1 + \pi x^2},$$

and then compute using the fundamental theorem of calculus and the substitution $u = x\sqrt{\pi}$:

$$0 \leq \int_{A_N} f d\mu \leq \int_{-N}^N \frac{1}{1 + \pi x^2} dx = \frac{2}{\sqrt{\pi}} \tan^{-1}(N\sqrt{\pi}) \leq \sqrt{\pi}.$$

This is a uniform upper bound on the nondecreasing sequence $\{\int_{A_N} f d\mu : N = 1, 2, \dots\}$, so the limit as $N \rightarrow \infty$ exists and is finite. Hence $f \in \mathcal{L}(\mathbf{R})$.

Now compute the integral using the trick from multivariable calculus:

$$\left(\int_{\mathbf{R}} f d\mu\right)^2 = \iint_{\mathbf{R}^2} e^{-\pi(x^2+y^2)} dx dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-\pi r^2} r dr d\theta = 2\pi \int_{r=0}^{\infty} e^{-\pi r^2} r dr,$$

using the change-of-variable formula $dx dy = r dr d\theta$ for polar coordinates, with $0 \leq r^2 = x^2 + y^2 < \infty$ and $0 \leq \theta \leq 2\pi$. Then substitute $u = \pi r^2$ to evaluate

$$\int_{r=0}^{\infty} e^{-\pi r^2} r dr = \lim_{N \rightarrow \infty} \int_{r=0}^N e^{-\pi r^2} r dr = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{u=0}^{\pi N^2} e^{-u} du = \lim_{N \rightarrow \infty} \frac{1}{2\pi} [1 - e^{-\pi N^2}] = \frac{1}{2\pi}.$$

Conclude that $\int_{\mathbf{R}} e^{-\pi x^2} d\mu = 1$. □