Ma 416: Complex Variables
Homework Assignment 11

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Due Thursday, December 1st, 2005

Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 4, sections 19A–20F.

1. Find an analytic function f whose real part is $\Re f(x+iy) = x^3y - xy^3$.

2. Find the conjugate harmonic function of $g(x,y) = e^{-y}\cos x$.

3. Is the function $h(x,y) = x^2+y^2$ the imaginary part of some function $f(x+iy)$ analytic in the unit disk in $\mathbf{C}$?

4. For $0 \le r < 1$ and $0 \le \phi \le 2\pi$, define the function

$$I(r,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,d\theta}{1+r^2-2r\cos(\theta-\phi)}.$$

   (This is the integral of the Poisson kernel $P(r,\theta-\phi)$.)

   (a) Show that $I(r,\phi)$ does not depend on ϕ. (Hint: substitute $\theta \leftarrow \theta' + \phi$.)

   (b) Show that $I(r,\phi)$ does not depend on r. (Hint: put $a = 2r/(1+r^2)$, observe that $(1-r^2)/(1+r^2) = \sqrt{1-a^2}$, and look up $\int_{-\pi}^{\pi} d\theta/(1-a\cos\theta) = 2\pi/\sqrt{1-a^2}$ in a table of integrals.)

   (c) Show that $I(r,\phi) = 1$ for all $0 \le r < 1$ and $0 \le \phi \le 2\pi$ by evaluating $I(0,0) = 1$ and using parts (a) and (b).

   (d) Conclude that if $u = u(x,y)$ is a harmonic function on the unit disk $D = \{x^2+y^2 \le 1\}$, and $u(x,y) = K$ for all $x^2+y^2 = 1$, then $u(x,y) = K$ for all $(x,y) \in D$.

5. Show directly that $u(x,y) = x^2 - y^2$ satisfies the averaging property: if $R > 0$, $C_R = \{r(\theta) = (x_0 + R\cos\theta, y_0 + R\sin\theta) : 0 \le \theta \le 2\pi\}$, and $ds = \|r'(\theta)\|\,d\theta$ is the arc length differential on C_R, then

$$\oint_{C_R} u(x,y)\,ds = 2\pi R\,u(x_0,y_0).$$

   How can Cauchy's integral formula be used to derive the same results?