Ma 416: Complex Variables
Solutions to Homework Assignment 7

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Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 2, sections 12A–13C.

1. Use the argument principle to count the zeros of $P(z) = z^4 + z^3 + 6z^2 + 3z + 5$ in the left half-plane $\{\Re z < 0\}$ and right half-plane $\{\Re z > 0\}$ of the complex plane.

   Solution: Since P has purely real and positive coefficients, it takes positive real values at all $z \in \mathbf{R}^+$. Check $P(0) = 5 \neq 0$ to conclude that P has no roots on $\mathbf{R}^+ \cup \{0\}$. Since the roots of a real-coefficient polynomial must come in conjugate pairs, P must have 0, 2, or 4 real roots on $\mathbf{R}^-$ with its remaining roots being nonreal conjugate pairs.

   Compute

$$\begin{aligned} P(z) &= z^4 + z^3 + 6z^2 + 3z + 5; \\ P'(z) &= 4z^3 + 3z^2 + 12z + 3; \qquad \text{at least one (negative) real root;} \\ P''(z) &= 12z^2 + 6z + 12; \qquad \text{nonreal roots } (1 \pm i\sqrt{15})/4; \\ P'''(z) &= 24z + 6; \qquad \text{real root } -1/4. \end{aligned}$$

   If P had four negative real roots, then by Rolle's theorem P' would have three and P'' would have two negative real roots, which is not the case. Hence P has either two or zero real roots in $\mathbf{R}^-$, which is a subset of the left half-plane $\{\Re z < 0\}$.

   Now localize the remaining two or four nonreal roots using the argument principle. Write

$$P(z) = z^4 \left(1 + \frac{1}{z} + \frac{6}{z^2} + \frac{3}{z^3} + \frac{5}{z^4}\right) \overset{\text{def}}{=} z^4 h(z); \qquad \Rightarrow \arg P(z) = \arg z^4 + \arg h(z).$$

   If $z = Re^{i\theta}$ for sufficiently large $R > 0$, then $h(z)$ will take values in the disk of radius $2/R$ centered at 1. Hence for any $\epsilon > 0$ we may take R large enough so that

$$|\arg P(z) - \arg(z^4)| = |\arg h(z)| < \epsilon; \qquad \Rightarrow \arg P(Re^{i\theta}) \approx 4\theta.$$

   For all $z = x \in \mathbf{R}$ we have $P(z) \in \mathbf{R}$, so we may take $\arg P(z) = 0$. For $z = iy \in i\mathbf{R}$, we have

$$\begin{aligned} P(iy) &= y^4 - iy^3 - 6y^2 + 3iz + 5; \\ P(iy) = 0 \;&\Rightarrow\; \Re P(iy) = y^4 - 6y^2 + 5 = 0 \text{ and } \Im P(iy) = -y^3 + 3y = 0. \end{aligned}$$

   But the real part has the roots ± 1 and $\pm\sqrt{5}$, while the imaginary part has different roots 0 and $\pm\sqrt{3}$. Hence P has no roots on the imaginary axis $\{\Re z = 0\}$.

We now compute the argument of $P(z)$ using inverse tangent:

$$\arg P(iy) = \arctan \frac{\Im P(iy)}{\Re P(iy)} = \arctan \frac{y(3-y^2)}{y^4 - 6y^2 + 5}.$$

As noted above, the real part in the denominator has four roots: $y = \pm 1$ and $y = \pm\sqrt{5}$. The imaginary part in the numerator has three roots: $y = 0$ and $y = \pm\sqrt{3}$. We may use this to determine which branches to use in order to have a continuous argument function $\theta = \arg P(z)$ along the positive imaginary axis $z = iy$:

y	$\tan\theta = \frac{\Im P(iy)}{\Re P(iy)}$	Quadrant of θ	$\theta = \arg P(iy)$
Near $+\infty$	$-/+ = -$	IV	Near 2π
$+\infty > y > \sqrt{5}$	$-/+ = -$	IV	$2\pi > \theta > 3\pi/2$
$y = \sqrt{5}$	$-/0 = -\infty$	IV to III	$3\pi/2$
$\sqrt{5} > y > \sqrt{3}$	$-/- = +$	III	$3\pi/2 > \theta > \pi$
$y = \sqrt{3}$	$0/- = 0$	III to II	π
$\sqrt{3} > y > 1$	$+/- = -$	II	$\pi > \theta > \pi/2$
$y = 1$	$+/0 = +\infty$	II to I	$\pi/2$
$1 > y > 0$	$+/+ = +$	I	$\pi/2 > \theta > 0$
$y = 0$	$0/+ = 0$	I	0

From this table we conclude that it is possible to define a continous function $\arg P(z)$ along the positively-oriented simple closed curve

$$C_I \stackrel{\text{def}}{=} [0, R] \cup C_R^+ \cup [iR, 0],$$

where C_R^+ is the quarter-circle from R to iR in Quadrant I, and R is sufficiently large. But then the argument principle implies that $P(z)$ has no zeros in Quadrant I.

Since the zeros of P come in complex conjugate pairs, the preceding argument also implies that P has no zeros in Quadrant IV. We conclude that P has no roots in the right half-plane $\{\Re z > 0\}$, and thus has four roots in the left half plane $\{\Re z < 0\}$. □

2. Use Rouché's theorem to determine the number of zeros of $3e^{z/2} + z$ satisfying $|z| < 1$.

Solution: Let $g(z) = z$ and $f(z) = 3e^{z/2}$, and let $C = \{z = \cos t + i \sin t\}$ be the positively oriented unit circle. We claim that $|g(z)| < |f(z)|$ on C, since

$$|f(z)| = 3|e^{(\Re z + i\Im z)/2}| = 3|e^{\Re z/2}| = 3e^{(\cos t)/2} \geq 3e^{-1/2} > 1,$$

while $|g(z)| = |z| = 1$. Thus Rouché's theorem applies: f and $f + g$ have the same number of zeros inside C. But $f(z)$ has no zeros anywhere, so there are no zeros of the function $f(z) + g(z) = 3e^{z/2} + z$ inside the circle C. □

3. Suppose $\{f_n : n = 1, 2, \ldots\}$ is an infinite sequence of analytic functions that converges uniformly in all compact subsets of a region D containing 0.

(a) Show that $\{\exp(f_n) : n = 1, 2, \ldots\}$ is also an infinite sequence of analytic functions that converges uniformly in each compact subset of D.

(b) Show that if $\lim_{n\to\infty} \exp(f_n(0)) = 0$, then $\lim_{n\to\infty} \exp(f_n(z)) = 0$ for all $z \in D$.

Solution: (a) First note that $\exp(g(z))$ is analytic whenever $g(z)$ is analytic, simply by using the chain rule: $[\exp(g(z))]' = g'(z)\exp(g(z))$. Both products and compositions of analytic functions are analytic.

Next, let $f(z) = \lim_{n\to\infty} f_n(z)$ for each $z \in D$. If $f_n \to f$ uniformly in some compact subset $K \subset D$, then $\exp(f_n) \to \exp(f)$ uniformly in K as well:

$$|\exp(f_n(z)) - \exp(f(z))| \leq C|f_n(z) - f(z)|, \qquad \text{all } z \in K,$$

where C is any constant larger than all values of $|f'(z)\exp(f(z))|$ in K.

(b) Since $\exp(f_n(z))$ has no zeros in D, and $\exp(f_n)$ converges uniformly in each compact subset of D, we may apply Hurwitz's theorem: either $\lim_{n\to\infty} \exp(f_n(z))$ is never zero for any $z \in D$, or else $\lim_{n\to\infty} \exp(f_n(z)) = 0$ for all $z \in D$. But $0 \in D$ and $\lim_{n\to\infty} \exp(f_n(0)) = 0$, so we conclude that $\lim_{n\to\infty} \exp(f_n(z)) = 0$ for all $z \in D$. □

4. Is it possible for a function $f = f(z)$ which takes only purely imaginary values to be analytic on $\{|z| < 1\}$?

Solution: The region $D = \{|z| < 1\}$ is an open set in $\mathbf{C}$, but no subset I of the imaginary axis can be an open set in $\mathbf{C}$. By the Open Mapping Theorem, if $f : D \to I$ is an analytic function, then it must be constant. Hence it is possible for analytic f to take only purely imaginary values on D, as long as f takes just one purely imaginary value. □

5. Show that $f(z) = z/(1-z)^2$ is univalent in $|z| < 1$.

Solution: Use direct computation. Let $D = \{z : |z| < 1\}$ be the open unit disk and suppose z, w belong to D with $f(z) = f(w)$. If $z = 0$ then $f(z) = 0$, so $f(w) = 0$, so $w = 0 = z$. Otherwise $z \neq 0$, so

$$\frac{z}{(1-z)^2} = \frac{w}{(1-w)^2} \Rightarrow w(1-z)^2 = z(1-w)^2 \Rightarrow zw^2 - (z^2+1)w + z = 0.$$

This may be regarded as a quadratic equation for w with coefficients determined by $z \neq 0$. Its roots are

$$w = \frac{z^2 + 1 \pm \sqrt{(z^2+1)^2 - 4z^2}}{2z} = \frac{z^2 + 1 \pm (z^2 - 1)}{2z} \in \{z, 1/z\}.$$

Of these two possible roots, $w = 1/z$ cannot satisfy $|w| < 1$ since $|z| < 1$. Hence we conclude that $w = z$. But $f(z) = f(w) \Rightarrow w = z$ for all $z, w \in D$ is the definition of univalence in D for f. □

6. Prove that the converse to Darboux's theorem is false: Find a simple closed curve S and an analytic function $f = f(z)$ such that f is univalent inside S but not univalent on S.

Solution: This is Exercise 13.5 on page 115 of our textbook. The function $f(z) = z^2$ is univalent in the open half-disk $D = \{z : |z| < 1, 0 < \arg(z) < \pi\}$, since the argument of z^2 will lie entirely within the principal range $(0, 2\pi)$. But D is bounded by the simple closed curve $S = \{e^{it} : 0 \leq t \leq \pi\} \cup \{t : -1 \leq t \leq 1\}$, and $z = -1 \in S$ and $z = 1 \in S$ both satisfy $z^2 = 1$. □