Ma 416: Complex Variables
Solutions to Homework Assignment 8

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Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 2, sections 14A–15F.

1. Find the Laurent series (in powers of $(z-0)$) in the punctured disk $0 < |z| < 1/4$ and in the annulus $1/4 < |z|$ for the function $f(z) = z^{-2}(4z-1)^{-1}$.

   Solution: In the punctured disk $0 < |z| < 1/4$, write

$$f(z) = \frac{1}{z^2(4z-1)} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-4^n)z^n = -16\sum_{n=-2}^{\infty} 4^n z^n.$$

   It is clear that the radius of convergence is $1/4$ and that there is a pole of order 2 at $z = 0$.

   In the annulus $|z| > 1/4$, write

$$f(z) = \frac{1}{z^2(4z-1)} = \frac{1/z^3}{(4-1/z)} = \frac{64}{[4z]^3}\frac{1}{4(1-1/[4z])} = \frac{16}{[4z]^3}\sum_{n=0}^{\infty}\frac{1}{[4z]^n} = \sum_{n=3}^{\infty}\frac{16}{[4z]^n}.$$

   It is clear that this series converges for all $|z| > 1/4$. □

2. Find three terms of the Maclaurin series for $f(z) = e^{-z}\sin z$, valid in some disk centered at zero.

   Solution: Multiply the first few terms of the Maclaurin series for the factor functions of $f(z)$, $e^{-z} = 1 - z + z^2/2 + \cdots$ and $\sin z = z - z^3/6 + \cdots$, to obtain the first three terms of the Maclaurin series for their product:

$$f(z) = (1\cdot z) + (-z\cdot z) + (1\cdot[-\frac{z^3}{6}] + \frac{z^2}{2}\cdot z) + \cdots = z - z^2 + \frac{z^2}{3} + \cdots,$$

   where the ellided terms are of degree 4 or higher. □

3. Find the Laurent series for $f(z) = e^z/(1-z)$ valid in a punctured neighborhood of ∞.

Solution: The Maclaurin series for e^z converges in all of $\mathbf{C}$, which contains every punctured neighborhood of ∞. Thus it suffices to find a Laurent series for $(1-z)^{-1}$ that converges in the complement of some disk and multiply the two series together. But

$$\frac{1}{1-z} = \frac{1}{z} \times \frac{-1}{1-1/z} = -\sum_{n=1}^{\infty} \frac{1}{z^n},$$

so

$$\frac{e^z}{1-z} = -\sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{n=1}^{\infty} \frac{1}{z^n} = -\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{z^{m-n}}{m!}.$$

This may be rearranged (since it converges absolutely in the complement of the unit disk) by substituting $j \leftarrow m-n$ and noting that the new summation ranges are $-\infty < j < \infty$ and $\max(0, j+1) \le m < \infty$:

$$\frac{e^z}{1-z} = -\sum_{j=-\infty}^{-1} \left(\sum_{m=0}^{\infty} \frac{1}{m!} \right) z^j - \sum_{j=0}^{\infty} \left(\sum_{m=j+1}^{\infty} \frac{1}{m!} \right) z^j = -e \sum_{j=-\infty}^{-1} z^j - \sum_{j=0}^{\infty} \left(e - \sum_{m=0}^{j} \frac{1}{m!} \right) z^j,$$

since $\sum_{m=0}^{\infty} 1/m! = e$ and $\sum_{m=j+1}^{\infty} 1/m! = e - \sum_{m=0}^{j} 1/m!$. □

4. Find three terms of the Laurent series for $f(z) = e^z / \sin z$ valid in some punctured disk centered at zero.

Solution: Since the function f has a pole of order 1 at $z = 0$, its Laurent series in a punctured disk centered at 0 will be of the form $f(z) = az^{-1} + b + cz + \cdots$. We compute the terms of degree $-1, 0, 1$. Note that

$$e^z = 1 + z + \frac{z^2}{2} + \cdots; \qquad \sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots,$$

so we may find equations for the undetermined coefficients a, b, c: $e^z = (\sin z) f(z)$, so

$$1 + z + \frac{z^2}{2} + \cdots = (z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots)(az^{-1} + b + cz + \cdots) = a + bz + (c - \frac{a}{6})z^2 + \cdots,$$

so $a = 1$, $b = 1$, and $c = 2/3$. This yields $f(z) = z^{-1} + 1 + \frac{2}{3}z + \cdots$. □

5. Use Laurent series to find the residue of $f(z) = z^{-6} e^{z^2} \tan z$ at $z = 0$.

Solution: Use the Maclaurin series

$$\begin{aligned} e^{z^2} &= \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} = 1 + z^2 + \frac{1}{2}z^4 + \frac{1}{6}z^6 + \cdots, \qquad \text{and} \\ \tan z &= \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(1 - 2^{2n})2^{2n}(-1)^n z^{2n-1} = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \cdots. \end{aligned}$$

Hence we may multiply these series together, and then multiply them by z^{-6}, to find the Laurent series for f. In fact, we only need the coefficient c_{-1} of z^{-1}, since that is the residue of f at $z = 0$. But that will be the coefficient of z^5 in the product $e^{z^2} \tan z$, which may be computed as follows:

$$(1 + z^2 + \frac{1}{2}z^4 + \cdots)(z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots) = \cdots + (\frac{2}{15} + \frac{1}{3} + \frac{1}{2})z^5 + \cdots,$$

so $c_{-1} = 29/30$ is the residue of f at 0. □

6. Find four terms in the Maclaurin series of $\sin(\sin z)$.

Solution: First note that

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \cdots,$$

so that the composition $\sin(\sin z)$ has expansion

$$(z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \cdots) - \frac{1}{6}(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots)^3 + \frac{1}{120}(z - \frac{z^3}{6} + \cdots)^5 + \cdots$$

But

$$\begin{aligned}(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots)^3 &= z^3 - 3\frac{z^5}{6} + 3\frac{z^7}{120} + \cdots;\\ (z - \frac{z^3}{6} + \cdots)^5 &= z^5 - 5\frac{z^7}{6} + \cdots,\end{aligned}$$

so

$$\begin{aligned}\sin(\sin z) &= z - (\frac{1}{6} + \frac{1}{6})z^3 + (\frac{1}{120} + \frac{1}{12} + \frac{1}{120})z^5 - (\frac{1}{5040} + \frac{1}{240} + \frac{1}{144})z^7 + \cdots\\ &= z - \frac{1}{3}z^3 + \frac{1}{10}z^5 - \frac{57}{5040}z^7 + \cdots\end{aligned}$$

□