# Ma 416: Complex Variables
# Solutions to Homework Assignment 10

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Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 3, sections 17A–18C.

1. Verify that $1/(1-z)$ can be continued outside the unit disk by expanding it about $z = ih$ for some $0 < h < 1$. Can you find an expansion about $z = i$?

   **Solution:** Write
   $$\frac{1}{1-z} = \frac{1}{[1-ih]-[z-ih]} = \left(\frac{1}{1-ih}\right)\frac{1}{1-\frac{z-ih}{1-ih}} = \left(\frac{1}{1-ih}\right)\sum_{n=0}^{\infty}\frac{(z-ih)^n}{(1-ih)^n}.$$
   This series has a radius of convergence
   $$R = \frac{1}{\lim_{n\to\infty}\sqrt[n]{1/|1-ih|^n}} = |1-ih| = \sqrt{1+h^2} > 1.$$
   The open disk of radius $R$ about $z = ih$ contains points such as $(1+h)i$ that are outside the unit disk centered at $z = 0$. Notice that it does not include $z = 1$.

   There is no obstruction to letting $h = 1$, or for that matter to letting $h = a$ for any $a \in \mathbf{R}$. □

2. Suppose $f(z) = \sum_{n=0}^{\infty} z^{2^n}$. Find the radius of convergence $R$ of this power series. Is there a function $g(z)$ analytic on a larger region than $D = \{|z| < R\}$ that agrees with $f(z)$ at all $z \in R$?

   **Solution:** First note that the function $f$ has power series $\sum_{k=0}^{\infty} a_{n_k} z^{n_k}$ for $n_k = 2^k$ and $a_{n_k} = 1$. Thus the radius of convergence is $R = 1/[\lim_{n\to\infty} \sqrt[2^n]{1}] = 1$.

   Then, since $n_{k+1}/n_k = 2 > 1$ for all $k$, Hadamard's gap theorem (section 17F, page 146 of our textbook) implies that $f$ has no analytic continuation outside $\{|z| < 1\}$. □

3. Use Abel's theorem to conclude that $\sum_{n=1}^{\infty}(-1)^n/n = -\ln 2$.

   **Solution:** The hypotheses are satisfied: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = -\log(1+z)$ with absolute convergence for all $|z| < 1$, and $A = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating sum theorem. Hence $\lim_{z\to 1-} -\log(1+z) = A$, where the convergence occurs along the positive real axis. But since $\log z = \ln z$ for positive real $z$, and ln is a continuous function, we conclude that $A = -\ln 2$. $\square$

4. Show that
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{8}} + \cdots$$
converges.

**Solution:** Rewrite the series as
$$\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}\right) + \left(\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}}\right) - \left(\frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}}\right) + \cdots$$
and re-label the terms as $\sum_{k=0}^{\infty} a_k$, where
$$a_k = (-1)^k \left(\frac{1}{\sqrt{2k+1}} + \frac{1}{\sqrt{2k+2}}\right).$$
But $a_k \to 0$ as $k \to \infty$, and the terms are strictly alternating in sign, so the series converges by Abel's convergence theorem. $\square$

5. Find the $(C, 1)$ sums of the series
   (a) $\sum_{n=0}^{\infty}(-1)^n$,
   (b) $\sum_{n=1}^{\infty}(-1)^n$, and
   (c) $1 - 1 + 0 + 1 - 1 + 0 + 1 - 1 + 0 + \cdots$ (where the terms $1, -1, 0$ repeat forever).

**Solution:** (a) The partial sums are
$$s_k = \sum_{i=0}^{k}(-1)^i = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd;} \end{cases}$$
The $(C, 1)$ sums $c_n = \frac{1}{n}\sum_{k=0}^{n-1} s_k$ therefore converge to $1/2$ as $n \to \infty$.

(b) The partial sums are
$$s_k = \sum_{i=1}^{k}(-1)^i = \begin{cases} 0, & \text{if } k \text{ is even,} \\ -1, & \text{if } k \text{ is odd;} \end{cases}$$
The $(C, 1)$ sums $c_n = \frac{1}{n}\sum_{k=0}^{n-1} s_k$ therefore converge to $-1/2$ as $n \to \infty$.

(c) Let $a_i$ denote the $i^{\text{th}}$ term of the sequence being summed, and suppose that the initial index is 0, so $a_0 = 1$. Denote by $k\%3$ the remainder after integer $k$ is divided by 3. Then the partial sums are
$$s_k = \sum_{i=0}^{k} a_i = \begin{cases} 1, & \text{if } k\%3 = 0, \\ 0, & \text{if } k\%3 = 1 \text{ or } k\%3 = 2; \end{cases}$$
The $(C, 1)$ sums $c_n = \frac{1}{n}\sum_{k=0}^{n-1} s_k$ therefore converge to $1/3$ as $n \to \infty$. $\square$

6. Show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent in $|x| < 1$ with $|na_n| \leq 8$ for all $n$, and $f(x) \to +\infty$ as $x \to 1-$, then $\sum_{n=0}^{\infty} a_n = +\infty$.

   **Solution:** This is similar to Exercise 18.13 on page 155 of our textbook. Let $Q > 0$ be given. Since $f(x) \to +\infty$ as $x \to 1-$, we may fix $x$ sufficiently close to 1 so that $f(x) = \sum_{n=0}^{\infty} a_n x^n > Q$.

   Note that for any integer $N \geq 0$, we may decompose

$$f(x) = \sum_{n=0}^{N} a_n x^n + \sum_{n=N+1}^{\infty} a_n x^n > Q \quad \Rightarrow \sum_{n=0}^{N} a_n x^n \geq Q - \left| \sum_{n=N+1}^{\infty} a_n x^n \right|.$$

   In particular, choosing $N \geq N_x = 1/(1-x) > 0$, we have the estimate

$$\left| \sum_{n=N+1}^{\infty} a_n x^n \right| \leq \sum_{n=N+1}^{\infty} |na_n| \frac{x^n}{n} < \frac{1}{N+1} \sum_{n=N+1} x^n < \frac{8}{N+1} \frac{1}{(1-x)} < \frac{8N}{N+1} < 8.$$

   For any $x < 1$, all sufficiently large $N \geq N_x$ satisfy this estimate. That leads to an estimate for $\sum_{n=0}^{\infty} a_n x_n$ that is independent of $x$:

$$(\forall x < 1)(\forall N \geq N_x) \sum_{n=0}^{N} a_n x^n > Q - 8 \quad \Rightarrow (\forall x < 1) \sum_{n=0}^{\infty} a_n x^n \geq Q - 8$$

   But then we may let $x \to 1-$ to conclude that $\sum_{n=0}^{\infty} a_n \geq Q - 8$. Since $Q$ was arbitrary, we conclude that $\sum_{n=0}^{\infty} a_n = +\infty$. □