Ma 450: Mathematics for Multimedia
Homework Assignment 4
Prof. Wickerhauser
Due Sunday, April 2nd, 2023

1. Fix $h > 0$. Given $y_-, y_+$, let $p = p(x)$ be the Lagrange polynomial through the points $(-h, y_-), (0, 0),$ and $(h, y_+)$.  
   (a) [6 points] Find a formula for the value $y = p(x)$ in terms of $h, x, y_-$, and $y_+$.  
   (b) [4 points] Find $p''(0)$ from the formula in part (a).

2. [10 points] Let $f(x) = x^2 + 1$ for $x \in [-1, 1]$. Find the expansion coefficients $c_0, c_1, c_2$ for $f$ in terms of Chebyshev polynomials $T_0(x), T_1(x), T_2(x)$, namely  
   $$f(x) = c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x).$$

3. Suppose $x_1 < x_2$, $y_1 < 0$, and $y_2 > 0$. Let $f$ be the piecewise linear function interpolating the set  
   $$\{(x_1, y_1), (x_2, y_2)\}.$$  
   (a) [5 points] On what interval (if any) is $f > 0$?  
   (b) [5 points] On what interval (if any) is $f < 0$?

4. Suppose that we have a machine that, given a random number $N$ of pennies, wraps them into bundles of 50, keeping 0 to 49 leftover pennies as its commission, and gives back $b(N)$ wrapped bundles. Let $50 \times b(N)$ be the estimate for the number of pennies $N$ measured by this “instrument.”  
   (a) [5 points] What is the quantization error of this instrument?  
   (b) [5 points] What is the imprecision?  
   (c) [5 points] What is the inaccuracy?  
   (d) [5 points] Is this instrument calibrated?

5. Let $f(x) = \cos(x) + 2 \sin(x)$ for $0 \leq x \leq 10$. Note that $f \in L^2([0, 10])$.  
   Let $x_k = k$ and $y_k = f(x_k)$ for $k = 0, 1, \ldots, 10$ be an interpolation set.  
   Estimate the signal-to-noise ratio in decibels for the following sampling approximations $s$ to $f$, using Octave and a grid of evaluation points in $[0, 10]$ with spacing 0.01:  
   (a) [10 points] The piecewise constant approximation using sampling function $1_{[-\frac{1}{2}, \frac{1}{2}]}$.  
   (b) [10 points] The piecewise linear approximation using the hat function.  
   (c) [10 points] The cubic spline approximation (so $s$ is the natural cubic spline defined by the interpolation set).
Hint: Compute \( \|f\|^2, \|s\|^2, \) and \( \|f-s\|^2 \) as sums of squares at the evaluation points \( \{0,0.01,0.02,\ldots,10\} \).

6. [10 points] Let \( f(x) = \cos(x) + 2\sin(x) \) for \( 0 \leq x \leq 10 \). Note that \( f \in L^2([0,10]) \).

Let \( s \) be the band-limited approximation to \( f \) with bandwidth 1, namely

\[
s(x) = \sum_{n=0}^{10} f(n) \text{sinc}(x-n).
\]

Estimate the signal-to-noise ratio in decibels for this approximation as in the previous problem.

Hint: Octave has a built-in \texttt{sinc()} function. Use it in your own function for \( s \), then compute \( \|f\|^2, \|s\|^2, \) and \( \|f-s\|^2 \) as sums of squares at the evaluation points \( \{0,0.01,0.02,\ldots,10\} \).

7. Let \( f = f(x,y) \) be the joint probability density supported on the region \( R = \{(x,y): 0 \leq x \leq 1, x-1 \leq y \leq x+1\} \) and defined by the formula \( f(x,y) = 1-|y-x| \) for \( (x,y) \in R \), with \( f(x,y) = 0 \) elsewhere.

(a) [5 points] Show that \( \int \int_R f(x,y) \, dx \, dy = 1 \).

(b) [5 points] Compute the normalizing constant \( c_x \) and determine \( f(y \mid x) \).

(c) [5 points] Compute the expectation \( E(y \mid x) \). Is \( d(x) = x \) an unbiased estimator?

(d) [5 points] Compute the risk \( R(d,y) \) for the decision function \( d(x) = x \). Does it depend on \( y \)?