Ma 450: Mathematics for Multimedia
Homework Assignment 5

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Due Sunday, April 16th, 2023

All solutions are worth 10 points.

1. Draw the graphs of $w(t)$, $w(t/2)$, and $w(3t)$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

2. Draw the graphs of $w(t - 3 - 4)$ and $w(t - 4 3)$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

3. Let $f = f(a) = f(a, b)$ be the function on Aff defined by $f(a) = 1_D(a)$, where $1_D$ is the indicator function of the region $D = \{a = (a, b) : A < a < A', B < b < B'\} \subset \text{Aff}$ for $0 < A < A'$ and $-\infty < B < B' < \infty$. Evaluate $\int_{\text{Aff}} f(a) \, da$ using the normalized left-invariant integral on Aff.

4. Let $w = w(t)$ be the Haar mother function and define

$$\phi_{M,K}^J(t) = \sum_{j=M+1}^{M+J} \frac{1}{2^j} w\left(\frac{t - K}{2^j}\right)$$

for arbitrary fixed $K \in \mathbb{R}$ and $M, J \in \mathbb{Z}$ with $J > 0$.

a. Show that

$$\lim_{J \to \infty} \phi_{M,K}^J(t) = 2^{-M} 1_{[K,K+2^M]}(t) \overset{\text{def}}{=} \phi_{M,K}(t),$$

b. Show that $\langle \phi_{M,K}^J, u \rangle \to \langle \phi_{M,K}, u \rangle$ as $J \to \infty$ for any function $u \in L^2(\mathbb{R})$.

(Hint: use Equation 5.4 and Lemma 5.1.)

5. Compute $\|w\|$, where

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log \xi)^2}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi \leq 0. \end{cases}$$

(Hint: use Plancherel’s theorem and Equation B.6 in Appendix B.)
6. Let $w$ be the function defined by
\[
\mathcal{F}w(\xi) = \begin{cases} 
    e^{-(\log |\xi|)^2}, & \text{if } \xi \neq 0; \\
    0, & \text{if } \xi = 0.
\end{cases}
\]
Show that $w$ is admissible and compute its normalization constant $c_w$.

7. Fix $A < 0$, $B > 0$, and $R > 1$ and suppose that $w = w(x)$ is a function satisfying $\mathcal{F}w(\xi) = 1$ if $RA < \xi < A$ or $B < \xi < RB$, with $\mathcal{F}w(\xi) = 0$, otherwise.
   a. Show that $w$ satisfies the admissibility condition of Theorem 5.2, and compute the normalization constant $c_w$.
   b. Give a formula for $w$.

8. Find a real-valued orthogonal low-pass CQF of length 4 satisfying the antisymmetry condition $h(0) = -h(3)$ and $h(1) = -h(2)$, or prove that none exist.

9. Find a real-valued orthogonal low-pass CQF of length 4 satisfying the symmetry condition $h(0) = h(3)$ and $h(1) = h(2)$, or prove that none exist.

10. Suppose that an orthogonal MRA has a scaling function $\phi$ satisfying $\phi(t) = 0$ for $t \notin [a,b]$. Prove that the low-pass filter $h$ for this MRA must satisfy $h(n) = 0$ for all $n \notin [2a-b, 2b-a]$. (This makes explicit the finite support of $h$ in Equation 5.36.)

11. Suppose that $h = \{h(k) : k \in \mathbb{Z}\}$ and $g = \{g(k) : k \in \mathbb{Z}\}$ satisfy the orthogonal CQF conditions. Show that the 2-periodizations $h_2, g_2$ of $h$ and $g$ are the Haar filters. Namely, show that $h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2}$.

12. Let $\phi$ be the scaling function of an orthogonal MRA, and let $\psi$ be the associated mother function. For $(x,y) \in \mathbb{R}^2$, define
\[
e_0(x,y) = \phi(x)\phi(y), \quad e_1(x,y) = \phi(x)\psi(y) \\
e_2(x,y) = \psi(x)\phi(y), \quad e_3(x,y) = \psi(x)\psi(y).
\]
Prove that the functions $\{e_n : n = 0, 1, 2, 3\}$ are orthonormal in $L^2(\mathbb{R}^2)$, the inner product space of square-integrable functions on $\mathbb{R}^2$. 
