1. Suppose that $f(t) = t^2$ if $-1 \leq t < 1$, and $f(t) = 0$ for all $t \notin [-1, 1)$. Find $f_1(t)$, the 1-periodization of $f$.

**Solution:** Since $f_1$ is 1-periodic, it suffices to find its values on one period interval, such as $[0, 1]$.
Since $f(t) = 0$ unless $t \in [-1, 1)$, the sum defining $f_1(t) = \sum_{k \in \mathbb{Z}} f(t + k)$ has exactly two nonzero terms. With the assumption that $t \in [0, 1)$, these are $f(t)$ and $f(t - 1)$. But then

$$f_1(t) = f(t) + f(t - 1) = t^2 + (t - 1)^2 = 2t^2 - 2t + 1.$$  

2. Define the reflection $R$ to be the transformation $Ru(t) \overset{\text{def}}{=} u(-t)$ acting on the vector space of functions of one real variable. Let $F$ be the fraying operator of Equation 3.14.

(a) Show that $R$ is a linear transformation.

(b) Find a formula for the compositions $RF$, $FR$, and $RFR$.

**Solution:** (a) $R$ is a linear transformation, since $R[au+bv](t) = au(-t) + bv(-t) = aRu(t) + bRv(t)$.

The compositions $RF$, $FR$, and $RFR$ are therefore linear transformations as well.

(b) Using Equation 3.14 gives the following formulas:

$$RFu(t) = \begin{cases} r(-t)u(-t) + r(t)u(t), & \text{if } -t > 0, \\ \tilde{r}(t)u(t) - \tilde{r}(-t)u(t), & \text{if } -t < 0, \end{cases}$$

$$FRu(t) = \begin{cases} r(t)Ru(t) + r(-t)Ru(-t), & \text{if } t > 0, \\ \tilde{r}(-t)Ru(t) - \tilde{r}(t)Ru(-t), & \text{if } t < 0, \end{cases}$$

$$RFRu(t) = \begin{cases} r(-t)Ru(-t) + r(t)Ru(t), & \text{if } -t > 0, \\ \tilde{r}(t)Ru(-t) - \tilde{r}(-t)Ru(t), & \text{if } -t < 0, \end{cases}$$

In all cases, $RFu(0) = FRu(0) = RFRu(0) = u(0)$.  

Due Sunday, March 5th, 2023
3. Show that the set of functions $\{\sqrt{2}\sin \pi nt : n = 1, 2, \ldots\}$ is orthonormal with respect to the inner product

$$\langle f, g \rangle \overset{\text{def}}{=} \int_{0}^{1} f(t)g(t) \, dt.$$ 

That is, show that

$$\langle \sqrt{2}\sin \pi nt, \sqrt{2}\sin \pi mt \rangle = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m, \end{cases}$$

for all $n, m \in \mathbb{Z}^+$.

**Solution:** First note that if $k$ is a nonzero integer, then

$$\int_{0}^{1} \cos \pi kt \, dt = \frac{1}{k\pi} [\sin k\pi \sin 0] = 0.$$ 

Then, recall the angle addition formulas for cosine: for any real numbers $A, B$,

$$\begin{align*}
\cos(A - B) &= \cos A \cos B + \sin A \sin B, \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B,
\end{align*}$$

so $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$. Thus the $n \neq m$ case yields

$$\langle \sqrt{2}\sin \pi nt, \sqrt{2}\sin \pi mt \rangle = 2 \int_{0}^{1} [\cos \pi(n - m)t - \cos \pi(n + m)t] \, dt = 0,$$

since both $n + m$ and $n - m$ are nonzero integers.

Finally, use the double angle formula $\cos 2A = 1 - 2\sin^2 A$ to compute

$$\langle \sqrt{2}\sin \pi nt, \sqrt{2}\sin \pi nt \rangle = 2 \int_{0}^{1} \sin^2 \pi nt \, dt = \int_{0}^{1} [1 - \cos 2\pi nt] \, dt = 1,$$

which gives the $n = m$ case. \(\square\)

4. Compute the sine-cosine Fourier series of the 1-periodic function $f(x) = \cos^2(5\pi x)$. (Hint: use a trigonometric identity.)

**Solution:** Since $\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos(2\theta)$, we have

$$\cos^2(5\pi x) = \frac{1}{2} + \frac{1}{2} \cos(5 \times 2\pi x).$$

Hence, the Fourier series

$$f(x) = a(0) + \sqrt{2} \sum_{n=1}^{\infty} [a(n) \cos(2\pi nx) + b(n) \sin(2\pi nx)]$$

has just two nonzero terms: $a(0) = \frac{1}{2}$ and $a(5) = \frac{1}{2 \sqrt{2}}$, with $a(n) = 0$ for all $n \notin \{0, 5\}$ and $b(n) = 0$ for all $n$. \(\square\)
5. Show that if \(|c(n)| < 2^{-|n|}\) for all integers \(n \neq 0\), then the 1-periodic function \(f = f(t)\) which is the inverse Fourier transform of the sequence \(\{c(n)\}\) must have a continuous \(d^{th}\) derivative for every positive integer \(d\).

**Solution:** This is a straightforward application of Corollary 3.10. Fix a positive integer \(d\) and note that for any positive integer \(N\),

\[
\sum_{k=-N}^{N} |k|^{d+1} |c(k)| < 2 \sum_{k=1}^{N} k^{d+1} 2^{-k} < 2 \sum_{k=1}^{\infty} k^{d+1} 2^{-k},
\]

and the infinite series converges by the ratio test, using the computation

\[
\lim_{k \to \infty} \frac{(k+1)^{d+1} 2^{-(k+1)}}{k^{d+1} 2^{-k}} = \frac{1}{2} \left( \lim_{k \to \infty} \frac{k+1}{k} \right)^{(d+1)} = \frac{1}{2} < 1.
\]

Thus \(f = \hat{c} \in \text{Lip}\) and also \(f^{(d)} \in \text{Lip}\), so \(f\) has \(d\) continuous derivatives. \(\square\)

6. Suppose that \(\phi\) has Fourier integral transform \(\mathcal{F}\phi\).

(a) Fix \(k \in \mathbb{R}\) and let \(\phi_k(x) \stackrel{\text{def}}{=} \phi(x + k)\). Show that \(\mathcal{F} \phi_k(\xi) = e^{2\pi ik \xi} \mathcal{F} \phi(\xi)\).

(b) Fix \(a > 0\) and let \(\phi_a(x) \stackrel{\text{def}}{=} \phi(ax)\). Show that \(\mathcal{F} \phi_a(\xi) = \frac{1}{a} \mathcal{F} \phi(\xi/a)\).

**Solution:** (a) Make the change of variable \(x \leftarrow y - k\), so \(dx \leftarrow dy\), and extract the factor \(e^{2\pi ik \xi}\):

\[
\mathcal{F} \phi_k(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i k \xi} \phi(x + k) \, dx = \int_{-\infty}^{\infty} e^{2\pi i k \xi} \phi(y) \, dy = e^{2\pi i k \xi} \mathcal{F} \phi(\xi).
\]

(b) Make the change of variable \(x \leftarrow y/a\), so \(dx \leftarrow \frac{1}{a} dy\), and extract the factor \(1/a\):

\[
\mathcal{F} \phi_a(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi} \phi(ax) \, dx = \frac{1}{a} \int_{-\infty}^{\infty} e^{-2\pi i \xi \left(\xi/a\right)} \phi(y) \, dy = \frac{1}{a} \mathcal{F} \phi(\xi/a).
\]

\(\square\)

7. Compute the inverse Fourier integral transform of the function

\[
\psi(\xi) = \begin{cases} 
1, & \text{if } -2 \leq \xi < -1 \text{ or } 1 < \xi \leq 2; \\
0, & \text{otherwise}.
\end{cases}
\]

(Hint: notice that \(\psi(\xi) = 1_I(\xi/4) - 1_I(\xi/2)\), where \(I = [-\frac{1}{2}, \frac{1}{2}]\).)

**Solution:** From Exercise 12, with \(f_a(x) \stackrel{\text{def}}{=} f(x/a)\), we get the identity

\[
\mathcal{F}^{-1} f_a(\xi) = \mathcal{F} f_a(-\xi) = a \mathcal{F} f(-a \xi) = a \mathcal{F}^{-1} f(a \xi).
\]

Thus, using the hint with \(a = 2\) or \(a = 4\) and the definition \(\mathcal{F} 1_I = \text{sinc}\), we calculate

\[
\mathcal{F}^{-1} \psi(x) = 4 \mathcal{F} 1_I(4x) - 2 \mathcal{F} 1_I(2x) = 4 \text{sinc } (4x) - 2 \text{sinc } (2x).
\]

\(\square\)
8. Compute the Fourier integral transform of the bump function

\[ b(x) = \begin{cases} 
2 - 2|x|, & \text{if } -1 \leq x < -\frac{1}{2} \text{ or } \frac{1}{2} < x \leq 1; \\
1, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2}; \\
0, & \text{otherwise.}
\end{cases} \]

**Solution:** First note that \( b(x) = 2h(x) - h(2x) \), where \( h \) is the hat function of Exercise 14 on p.105 in the textbook:

\[ h(x) = \begin{cases} 
1 - |x|, & \text{if } -1 \leq x \leq 1; \\
0, & \text{otherwise.}
\end{cases} \]

But then \( \mathcal{F}b(\xi) = 2\mathcal{F}h(\xi) - \frac{1}{2}\mathcal{F}h(\xi/2) = 2\text{sinc}(\xi)^2 - \frac{1}{2} [\text{sinc}(\xi/2)]^2 \), using the results of Exercises 12 and 14 in section 3.3 of the textbook.

9. Show that the vectors \( \vec{\omega}_n \in \mathbb{C}^N, n = 0, 1, \ldots, N - 1 \) defined by \( \vec{\omega}_n(k) = \exp(-2\pi i nk/N) \) form an orthonormal basis with respect to the inner product

\[ \langle f, g \rangle \overset{\text{def}}{=} \frac{1}{N} \sum_{k=0}^{N-1} f(k) g(k). \]

**Solution:** Use the geometric series summation formula:

\[ \frac{1}{N} \sum_{k=0}^{N-1} \vec{\omega}_n(k) \vec{\omega}_m(k) = \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i nk/N} e^{-2\pi i mk/N} = \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i (n-m)k/N} = \begin{cases} 
\frac{1}{N}, & \text{if } n \neq m; \\
\frac{1}{N} e^{-2\pi i (n-m)/N}, & \text{if } n = m,
\end{cases} = \delta(n-m). \]

Orthonormality implies linear independence, and any \( N \) linearly independent vectors in an \( N \)-dimensional vector space must be a basis.

10. Write out explicitly the matrices for the \( 3 \times 3 \) discrete inverse Fourier and Hartley transforms (\( F^{-1}_3 \) and \( H^{-1}_3 \)).

**Solution:** The normalized formulas \( F^{-1}_N(m,n) = \frac{1}{\sqrt{N}} \exp(2\pi i mn/N) \) and \( H^{-1}_N(m,n) = H_N(m,n) = \frac{1}{\sqrt{N}} \left[ \cos(2\pi mn/N) + \sin(2\pi mn/N) \right] \) give:

\[ F^{-1}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 + i\sqrt{3} & -1 - i\sqrt{3} \\ 1 & -1 - i\sqrt{3} & -1 + i\sqrt{3} \end{pmatrix}; \]

\[ H^{-1}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 + \sqrt{3} & -1 - \sqrt{3} \\ 1 & -1 - \sqrt{3} & -1 + \sqrt{3} \end{pmatrix}. \]
11. What is the matrix \((C_N^{IV})^2\) of the square of \(N \times N\) DCT-IV? Give a formula for every positive integer \(N\).

Solution: Since \(\sqrt{\frac{2}{N}} C_N^{IV}\) is a symmetric unitary transformation, it is its own inverse:

\[
\left( \sqrt{\frac{2}{N}} C_N^{IV} \right)^{-1} = \left( \sqrt{\frac{2}{N}} C_N^{IV} \right)^* = \sqrt{\frac{2}{N}} C_N^{IV}
\]

\[
\Rightarrow \left( \sqrt{\frac{2}{N}} C_N^{IV} \right)^2 = Id \quad \Rightarrow \quad (C_N^{IV})^2 = \frac{N}{2} Id.
\]

This is a direct consequence of Theorem 3.15. \(\square\)