All solutions are worth 10 points.

1. Draw the graphs of $w(t)$, $w(t/2)$, and $w(3t)$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

**Solution:** The graphs are shown in Figure 1.

2. Draw the graphs of $w(t - 4)$ and $w(t - 4)$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

**Solution:** The graphs are shown in Figure 2.

3. Let $f = f(a) = f(a, b)$ be the function on $\text{Aff}$ defined by $f(a) = 1_D(a)$, where $1_D$ is the indicator function of the region $D = \{a = (a, b) : A < a < A', B < b < B'\} \subset \text{Aff}$ for $0 < A < A'$ and $-\infty < B < B' < \infty$. Evaluate $\int_{\text{Aff}} f(a) \, da$ using the normalized left-invariant integral on $\text{Aff}$.

Figure 1: Graphs of $w(t)$, $w(t/2)$, and $w(3t)$ for the Haar function $w$. 

Figure 2: Graphs of $w(t - 4)$ and $w(t - 4)$ for the Haar function $w$. 

1
Figure 2: Graphs of $w(\frac{t}{3} - 4)$ and $w(\frac{t-4}{3})$ for the Haar function $w$.

Solution: Use the integral defined in Equation 5.19:

$$\int_{\text{Aff}} f(a) \, da \overset{\text{def}}{=} \int_{a=-\infty}^{\infty} \int_{b=0}^{\infty} f(a, b) \frac{dadb}{a^2} = \int_{a=-B}^{\infty} \int_{a=A}^{A'} \frac{1}{a^2} \, dadb$$

$$= (B' - B) \int_{a=A}^{A'} \frac{1}{a^2} \, da = (B' - B) \left( \frac{1}{A} - \frac{1}{A'} \right).$$

4. Let $w = w(t)$ be the Haar mother function and define

$$\phi_{M,K}^J(t) \overset{\text{def}}{=} \sum_{j=M+1}^{M+J} \frac{1}{2^j} w \left( \frac{t - K}{2^j} \right)$$

for arbitrary fixed $K \in \mathbb{R}$ and $M, J \in \mathbb{Z}$ with $J > 0$.

a. Show that

$$\lim_{J \to \infty} \phi_{M,K}^J(t) = 2^{-M} 1_{[K, K+2^M)}(t) \overset{\text{def}}{=} \phi_{M,K}(t),$$

b. Show that $\langle \phi_{M,K}^J, u \rangle \to \langle \phi_{M,K}, u \rangle$ as $J \to \infty$ for any function $u \in L^2(\mathbb{R})$.

(Hint: use Equation 5.4 and Lemma 5.1.)

Solution:

a. Using $\phi^J$ as defined in Equation 5.4, evaluate

$$\phi_{M,K}^J(t) = \sum_{j=M+1}^{M+J} \frac{1}{2^j} w \left( \frac{t - K}{2^j} \right) = \sum_{j=1}^{J} \frac{1}{2^{j+M}} w \left( \frac{t - K}{2^{j+M}} \right)$$

$$= \frac{1}{2^M} \sum_{j=1}^{J} \frac{1}{2^j} w \left( \frac{1}{2j} \frac{t - K}{2^M} \right) = \frac{1}{2^M} \phi^J \left( \frac{t - K}{2^M} \right).$$
Thus for all $t \in \mathbb{R}$,

$$\lim_{J \to \infty} \phi_{M,K}^J(t) = \frac{1}{2M} \lim_{J \to \infty} \phi^J \left( \frac{t - K}{2M} \right) = \frac{1}{2M} 1 \left( \frac{t - K}{2M} \right) = 2^{-M} 1_{\{K + 2M\}}(t).$$

b. Lemma 5.1 then gives

$$\lim_{J \to \infty} \langle \phi_{M,K}^J, u \rangle = \frac{1}{2M} \int_{-\infty}^{\infty} \left( \frac{t - K}{2M} \right) u(t) dt = \frac{1}{2M} \int_{K}^{K+2M} u(t) dt = \langle \phi_{M,K}, u \rangle,$$

as claimed.

5. Compute $\|w\|$, where

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log \xi)^2}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi \leq 0. \end{cases}$$

(Hint: use Plancherel’s theorem and Equation B.6 in Appendix B.)

**Solution:** By Plancherel’s theorem, $\|w\| = \|\mathcal{F}w\|$. Substitute $\xi \leftarrow e^{\eta + \frac{1}{2}}$ to compute

$$\|\mathcal{F}w\|^2 = \int_{\xi=0}^{\infty} e^{-2(\log \xi)^2} d\xi = \int_{\eta=-\infty}^{\infty} e^{-2(\eta+\frac{1}{2})^2} e^{\eta+\frac{1}{2}} d\eta = e^{1/8} \int_{\eta=-\infty}^{\infty} e^{-2\eta^2} d\eta.$$

Finally, substitute $x \leftarrow \eta \sqrt{\frac{\pi}{2}}$ into Equation B.6, $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$, to get $\int_{-\infty}^{\infty} e^{-2\eta^2} d\eta = \sqrt{\frac{\pi}{2}}$. Thus $\|w\| = \sqrt{\|\mathcal{F}w\|^2} = e^{1/16} \left( \frac{\pi}{2} \right)^{1/4} \approx 1.1917$.

NOTE: The norm may also be computed with Macsyma as follows:

```
sqrt(integrate(exp(-log(x)**2)**2,x,0,inf));
```

A numerical approximation may then be found using the `float()` command.

6. Let $w$ be the function defined by

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log |\xi|)^2}, & \text{if } \xi \neq 0; \\ 0, & \text{if } \xi = 0. \end{cases}$$

Show that $w$ is admissible and compute its normalization constant $c_w$.

**Solution:** First note that $\mathcal{F}w(-\xi) = \mathcal{F}w(\xi)$, so $|\mathcal{F}w(-\xi)|^2 = |\mathcal{F}w(\xi)|^2$. Thus if the $+\xi$ admissibility integral exists, then the $-\xi$ integral exists as well and has the same value.
Next, compute the $+\xi$ admissibility integral:

$$c_w = \int_0^\infty \frac{|Fw(\xi)|^2}{\xi} d\xi = \int_\xi=0^\infty e^{-2(\log \xi)^2} d\xi = \int_\eta=-\infty^{\infty} e^{-2\eta^2} e^{\eta} d\eta = \int_\eta=-\infty^{\infty} e^{-2\eta^2} d\eta.$$  

This follows from the substitution $\xi \leftrightarrow e^\eta$. Finally, substitute $x \leftrightarrow \eta \sqrt{\frac{2}{\pi}}$ into Equation B.6, $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$, to get $c_w = \int_{-\infty}^{\infty} e^{-2\eta^2} d\eta = \sqrt{\frac{\pi}{2}} \approx 1.2533$. Thus $w$ is admissible.

**NOTE:** This integral may also be computed with Macsyma:

integrate(exp(-log(x)**2)**2/x,x,0,inf);

A numerical approximation may then be found using the float() command.  

7. Fix $A < 0$, $B > 0$, and $R > 1$ and suppose that $w = w(x)$ is a function satisfying $Fw(\xi) = 1$ if $RA < \xi < A$ or $B < \xi < RB$, with $Fw(\xi) = 0$, otherwise.

a. Show that $w$ satisfies the admissibility condition of Theorem 5.2, and compute the normalization constant $c_w$.

b. Give a formula for $w$.

**Solution:**

a. Plancherel’s theorem guarantees that $w$ belongs to $L^2(\mathbb{R})$, since $\|w\| = \|Fw\| = \sqrt{(B-A)(R-1)} < \infty$.

Compute the two admissibility integrals:

$$\int_0^\infty \frac{|Fw(-\xi)|^2}{\xi} d\xi = \int_{-A}^{-RA} \frac{1}{\xi} d\xi = \log(-RA) - \log(-A) = \log R;$$

$$\int_0^\infty \frac{|Fw(\xi)|^2}{\xi} d\xi = \int_{B}^{RB} \frac{1}{\xi} d\xi = \log(RB) - \log(B) = \log R.$$  

These are finite and equal for $R > 1$, so $w$ is admissible with normalization constant $c_w = \log R$.

b. The inverse Fourier integral transform of $Fw$ gives the formula for $w$:

$$w(x) = \int_{-\infty}^{\infty} e^{2\pi i \xi} Fw(\xi) d\xi = \int_{RA}^{A} + \int_{B}^{RB} e^{2\pi i \xi} d\xi = \frac{e^{2\pi i x A} - e^{2\pi i x RA} + e^{2\pi i x RB} - e^{2\pi i x B}}{2\pi i x}.$$  

Note that $w(0) \overset{\text{def}}{=} (B-A)(R-1)$ makes the function continuous, in fact smooth, by L'Hôpital’s rule. Further simplification is not necessary.  

4
8. Find a real-valued orthogonal low-pass CQF of length 4 satisfying the antisymmetry condition \( h(0) = -h(3) \) and \( h(1) = -h(2) \), or prove that none exist.

Solution: None exist. Antisymmetry would violate the normalization condition \( h(0) + h(2) = \frac{1}{\sqrt{2}} = h(1) + h(3) \). Thus no antisymmetric real-valued orthogonal low-pass CQFs of length 4 exist. \( \square \)

9. Find a real-valued orthogonal low-pass CQF of length 4 satisfying the symmetry condition \( h(0) = h(3) \) and \( h(1) = h(2) \), or prove that none exist.

Solution: Only one exists and it is degenerate in that it only has 2 nonzero coefficients. Let \( h \) be an orthogonal CQF with nonzero real coefficients \( h(0), h(1), h(2), h(3) \). Then \( h \) must be of the form

\[
\begin{align*}
    h(0) &= \frac{1 - c}{\sqrt{2}(1 + c^2)}; \\
    h(1) &= \frac{1 + c}{\sqrt{2}(1 + c^2)}; \\
    h(2) &= \frac{c(c + 1)}{\sqrt{2}(1 + c^2)}; \\
    h(3) &= \frac{c(c - 1)}{\sqrt{2}(1 + c^2)},
\end{align*}
\]

where \( c \) is some real number different from 0 and \( \pm 1 \). The symmetry conditions imply \( 1 - c = c(c - 1) \) and \( 1 + c = c(c + 1) \), which implies \( c^2 = 1 \) and thus \( c = \pm 1 \). But \( c = 1 \) leads to a filter of length 2, the Haar filter. The other choice \( c = -1 \) leads to the unique symmetric real-valued orthogonal CQF of length 4: \( \{1/\sqrt{2}, 0, 0, 1/\sqrt{2}\} \). \( \square \)

10. Suppose that an orthogonal MRA has a scaling function \( \phi \) satisfying \( \phi(t) = 0 \) for \( t \notin [a, b] \). Prove that the low-pass filter \( h \) for this MRA must satisfy \( h(n) = 0 \) for all \( n \notin [2a - b, 2b - a] \). (This makes explicit the finite support of \( h \) in Equation 5.36.)

Solution: If \( \phi(t) = 0 \) for \( t \notin [a, b] \), then \( \phi(2t - n) = 0 \) for \( t \notin \left[\frac{a + n}{2}, \frac{b + n}{2}\right] \). Use the orthonormality of \( \{\sqrt{2}\phi(2t - k) : k \in \mathbb{Z}\} \) in \( V_{-1} \) to compute

\[
\frac{1}{\sqrt{2}} h(n) = \left\langle \phi(2t - n), \sum_k h(k) \sqrt{2} \phi(2t - k) \right\rangle = \left\langle \phi(2t - n), \phi(t) \right\rangle.
\]

But the support intervals \( \left[\frac{a + n}{2}, \frac{b + n}{2}\right] \) and \( [a, b] \) of the two factors in the inner product will not overlap if \( (a + n)/2 > b \iff n > 2b - a \) or if \( (b + n)/2 < a \iff n < 2a - b \). Thus \( h(n) = 0 \) if \( n \notin [2a - b, 2b - a] \). \( \square \)

11. Suppose that \( h = \{h(k) : k \in \mathbb{Z}\} \) and \( g = \{g(k) : k \in \mathbb{Z}\} \) satisfy the orthogonal CQF conditions. Show that the 2-periodizations \( h_2, g_2 \) of \( h \) and \( g \) are the Haar filters. Namely, show that \( h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2} \).
Solution: Use the normalization conditions for \( h \) and \( g \) to evaluate the 2-periodization formula:

\[
h_2(0) = \sum_k h(0 + 2k) = 1/\sqrt{2}; \quad h_2(1) = \sum_k h(1 + 2k) = 1/\sqrt{2};
\]

\[
g_2(0) = \sum_k g(0 + 2k) = 1/\sqrt{2}; \quad g_2(1) = \sum_k g(1 + 2k) = -1/\sqrt{2}.
\]

Since \( h_2 \) and \( g_2 \) are 2-periodic, this determines all their values. \( \square \)

12. Let \( \phi \) be the scaling function of an orthogonal MRA, and let \( \psi \) be the associated mother function. For \((x, y) \in \mathbb{R}^2\), define

\[
e_0(x, y) = \phi(x)\phi(y), \quad e_1(x, y) = \phi(x)\psi(y)
\]

\[
e_2(x, y) = \psi(x)\phi(y), \quad e_3(x, y) = \psi(x)\psi(y).
\]

Prove that the functions \( \{e_n : n = 0, 1, 2, 3\} \) are orthonormal in \( L^2(\mathbb{R}^2) \), the inner product space of square-integrable functions on \( \mathbb{R}^2 \).

Solution: First note that \( \phi \) and \( \psi \) are compactly supported, so \( e_0, e_1, e_2, e_3 \) vanish outside some closed and bounded rectangles in \( \mathbb{R}^2 \). Also, both \( \phi \) and \( \psi \) are integrable and square-integrable as functions of one variable, so \( e_0, e_1, e_2, e_3 \) are integrable by iterating one-dimensional integrals. Write \( e_i^1(x) \) for the \( x \)-dependent factor of \( e_i(x, y) \) and \( e_i^2(y) \) for the \( y \)-dependent factor of \( e_i(x, y) \); then \( e_0^1 = e_1^2 = e_0^2 = e_1^2 = \phi \), while \( e_2^1 = e_3^1 = e_2^2 = e_3^2 = \psi \). Thus the inner products in \( L^2(\mathbb{R}^2) \) are computable as follows:

\[
\langle e_i, e_j \rangle = \iint_{\mathbb{R}^2} e_i(x, y)e_j(x, y) \, dx \, dy
\]

\[
= \left( \int_{\mathbb{R}} e_i^1(x) \, dx \right) \left( \int_{\mathbb{R}} e_j^2(y) \, dy \right)
\]

\[
= \langle e_i^1, e_j^1 \rangle \langle e_i^2, e_j^2 \rangle, \quad i, j \in \{0, 1, 2, 3\}.
\]

But if \( i \neq j \), then at least one of these inner products is \( \langle \phi, \psi \rangle \) or \( \langle \psi, \phi \rangle \), which are both zero since \( \phi \perp \psi \). Hence \( \{e_0, e_1, e_2, e_3\} \) is an orthogonal set in \( L^2(\mathbb{R}^2) \).

On the other hand, if \( i = j \), then the two inner products are both 1 since \( \|\phi\| = \|\psi\| = 1 \). Hence \( \{e_0, e_1, e_2, e_3\} \) is an orthonormal set in \( L^2(\mathbb{R}^2) \). \( \square \)