Math 456: Introduction to Financial Mathematics

Homework Set 1

1. Suppose that a risky asset S has spot price $S(0) = 100$ and that the riskless return to $T = 0.5$ year is $R = 1.0237$. Assuming there are no arbitrages, compute the following:

   (a) the current zero-coupon bond discount $Z(0, T)$,

   (b) the Forward price for one unit of S at expiry T,

   (c) the riskless annual interest rate (assuming continuous compounding),

2. With $S(0)$, R, and T as in Exercise 1, suppose that $S(T)$ is modeled by

$S(T)$	93	95	98	104	107	110	114
$\Pr(S(T))$	0.02	0.05	0.09	0.29	0.31	0.19	0.05

   (a) What is the expected future price $S(T)$ in this finite probability space model?

   (b) Use this finite probability space model to estimate premiums $C(0)$ and $P(0)$ for European-style Call and Put options, respectively, with strike price $K = 101$ and expiry T.

   (c) Does Call-Put Parity hold in this model? What might cause it to be inaccurate?

3. Use the no arbitrage Axiom 1 to prove that Eq.1.7 holds.

4. Prove Corollary 1.3. (Hint: review the proof of Theorem 1.2.)

5. Suppose, in contradiction with Eq.1.16, that $C(0) - P(0) < S(0) - K/R$. Construct an arbitrage.

6. Prove Eq.1.20, the Call-Put parity Formula for foreign exchange options:

   $$C(0) - P(0) = \frac{X(0)}{R_f} - \frac{K}{R_d},$$

   using the no arbitrage axiom.

7. (a) Prove that the plus-part function satisfies Eq.1.17:

   $$[X]^+ - [-X]^+ = X,$$

   for any number X.

(b) Apply the identity in part (a) to the payoff values of European-style Call and Put options for S at strike price K and expiry T to show Eq.1.18:

$$C(T) - P(T) = S(T) - K.$$

8. Plot the payoff and profit graphs for the following colorfully named option portfolios as a function of the price $S(T)$ at expiry time T:

   (a) *Long straddle:* buy one Call and one Put on S with the same expiry T and at-the-money strike price $K \approx S(0)$. For what values of $S(T)$ will this be profitable?

   (b) *Long strangle:* buy one Call at K_c and one Put at K_p with the same expiry T but with out-of-the-money strike prices $K_p < S(0) < K_c$. How does its profitability compare with that of a long straddle?

9. A *butterfly spread* is a portfolio of European-style Call options purchased at time $t = 0$ with the same expiry $t = T$ but at three strike prices $L < M < H$, where $M = \frac{1}{2}(L + H)$:

   - buy one Call C_L at strike price L for $C_L(0)$,
   - buy one Call C_H at strike price H for $C_H(0)$,
   - sell two Calls C_M short at strike price M for $2C_M(0)$.

   (a) Plot the payoff graph for the butterfly spread at expiry when its price is $C_L(T) + C_H(T) - 2C_M(T)$. Mark the three strike prices on the $S(T)$ axis.

   (b) Conclude from the graph for (a) that $C_M(0) < \frac{1}{2}[C_L(0) + C_H(0)]$. (Hint: use the no arbitrage Axiom 1.)

10. An *iron condor* is a portfolio $C_1 - C_2 - P_3 + P_4$ of four European-style options. To construct it, simultaneously buy one Call at K_1, sell one Call at K_2, sell one Put at K_3, and buy one Put at K_4, all with the same expiry T but with $K_1 < K_2 < K_3 < K_4$.

    (a) Plot or describe the payoff graph for an iron condor portfolio at expiry.

    (b) Assuming no arbitrage, prove that the portfolio must have a positive net premium.

    (c) Assuming no arbitrage, find inequalities bounding the maximum profit and the maximum loss of an iron condor portfolio at expiry.