Math 456: Introduction to Financial Mathematics

Homework Set 2

1. Let
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$
be the cumulative distribution function of the standard normal random variable. Prove that
$$\Phi(x) + \Phi(-x) = 1$$
for every x.

2. Rewrite the recursive definition of the random walk X in Section 2.2 with these normalizations:
   - Multiply by $1/\sqrt{n}$ to have variance 1.
   - Change the time step to $1/n$ so that the time interval is $[0, 1]$.

3. Derive the Black-Scholes formula for European-style Put options, Eq.2.26:
$$P(0) = e^{-rT} K \Phi(-d_2) - S_0 \Phi(-d_1).$$
(Hint: follow the steps in Section 2.3.1, but use the Put payoff $[K - S(T)]^+$ at expiry.)

4. Let d_1, d_2 be defined as in Eq.2.24, and let ϕ be the standard normal p.d.f. defined in Eq.2.18. Show that
$$S_0 \phi(d_1) - K e^{-rT} \phi(d_2) = 0.$$
(Hint: Notice that $d_1 - d_2 = v\sqrt{T}$ and
$$d_1 + d_2 = 2(\log \frac{S_0}{K} + rT)/(v\sqrt{T}),$$
and thus $(d_1^2 - d_2^2)/2 = (d_1 - d_2)(d_1 + d_2)/2 = \log(S_0/K) + rT$.)

5. Derive Δ_C and Δ_P in Section 2.3.3 by differentiating the Black-Scholes formulas.

6. Derive Γ_C and Γ_P in Section 2.3.3 from Black-Scholes.

7. Derive Θ_C and Θ_P in Section 2.3.3 from Black-Scholes.

8. Derive κ_C and κ_P in Section 2.3.3 from Black-Scholes.

9. Derive ρ_C and ρ_P in Section 2.3.3 from Black-Scholes.

10. Implement the computation of all the Black-Scholes Greeks in Octave or MATLAB and add this functionality to the program `BS()` in Section 2.4. Apply it to compute the premiums and all Greeks for European-style Call and Put options on a risky asset with the following parameters: spot price \$91, strike price \$94, expiry in 0.5 years, annual riskless rate 4%, volatility 20%.

11. (a) Verify Eq.2.34 relating Θ_C, Δ_C, and Γ_C.

    (b) Find the equivalent relation for Puts and verify it.

12. (a) Find the coefficients p_1, p_2, p_3 that give the least-squares best fit

$$f(x) = p_1 + p_2 e^x + p_3 e^{-x}$$

    to the data $\{(x, y)\} = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$.

    (b) Plot f at 81 equispaced points on a graph showing the data.

13. At one instant, two days before expiry, near-the-money Call options for ABC common stock had the following prices:

Strike price	Premium	Open interest
44.00	3.80	3,260
45.00	2.77	4,499
46.00	1.77	3,862
47.00	0.78	6,271
48.00	0.18	10,156
49.00	0.03	10,619
50.00	0.01	14,219

    The spot price for ABC is \$47.58. Estimate the premium for the at-the-money Call option in the following ways:

    (a) Unweighted quadratic regression.

    (b) Weighted quadratic regression.

    (c) Polynomial interpolation.

    (d) Spline interpolation.