Math 456: Introduction to Financial Mathematics

Homework Set 2

Due 2 October 2022

1. Let

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \]

be the cumulative distribution function of the standard normal random variable. Prove that

\[ 1 - \Phi(x) = \Phi(-x) \]

for every \( x \).

2. Rewrite the recursive definition of the random walk \( X \) in Section 2.2 with these normalizations:
   - Multiply by \( 1/\sqrt{n/4} = 2/\sqrt{n} \) to have variance 1.
   - Change the time step to \( 1/n \) so that the time interval is \([0, 1]\).

3. Derive the Black-Scholes formula for European-style Put options, Eq.2.26:

\[ P(0) = e^{-rT} K \Phi(-d_2) - S_0 \Phi(-d_1). \]

(Hint: follow the steps in Section 2.3.1, but use the Put payoff \([K - S(T)]^+\) at expiry.)

4. Let \( d_1, d_2 \) be defined as in Eq.2.24, and let \( \phi \) be the standard normal p.d.f. defined in Eq.2.18. Show that

\[ S_0 \phi(d_1) - Ke^{-rT} \phi(d_2) = 0. \]

(Hint: Notice that \( d_1 - d_2 = v\sqrt{T} \) and

\[ d_1 + d_2 = 2(\log \frac{S_0}{K} + rT)/(v\sqrt{T}), \]

and thus \( (d_1^2 - d_2^2)/2 = (d_1 - d_2)(d_1 + d_2)/2 = \log(S_0/K) + rT. \)

5. Derive \( \Delta_C \) and \( \Delta_P \) in Section 2.3.3 by differentiating the Black-Scholes formulas.

6. Derive \( \Gamma_C \) and \( \Gamma_P \) in Section 2.3.3 from Black-Scholes.
7. Derive $\Theta_C$ and $\Theta_P$ in Section 2.3.3 from Black-Scholes.

8. Derive $\kappa_C$ and $\kappa_P$ in Section 2.3.3 from Black-Scholes.

9. Derive $\rho_C$ and $\rho_P$ in Section 2.3.3 from Black-Scholes.

10. Implement the computation of all the Black-Scholes Greeks in Octave or MATLAB and add this functionality to the program $\text{BS}(\cdot)$ in Section 2.4. Apply it to compute the premiums and all Greeks for European-style Call and Put options on a risky asset with the following parameters: spot price $90$, strike price $95$, expiry in 1 year, annual riskless rate 2%, volatility 15%.

11. (a) Verify Eq.2.34 relating $\Theta_C$, $\Delta_C$, and $\Gamma_C$.

(b) Find the equivalent relation for Puts and verify it.

12. (a) Find the coefficients $p_1, p_2, p_3$ that give the least-squares best fit
     
     $$f(x) = p_1 + p_2 e^x + p_3 e^{-x}$$

     to the data $\{(x, y)\} = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$.

(b) Plot $f$ at 81 equispaced points on a graph showing the data.

13. At one instant, two days before expiry, near-the-money Call options for ABC common stock had the following prices:

<table>
<thead>
<tr>
<th>Strike price $S$</th>
<th>Premium $P$</th>
<th>Open interest $O$</th>
</tr>
</thead>
<tbody>
<tr>
<td>44.00</td>
<td>3.80</td>
<td>3,260</td>
</tr>
<tr>
<td>45.00</td>
<td>2.77</td>
<td>4,499</td>
</tr>
<tr>
<td>46.00</td>
<td>1.77</td>
<td>3,862</td>
</tr>
<tr>
<td>47.00</td>
<td>0.78</td>
<td>6,271</td>
</tr>
<tr>
<td>48.00</td>
<td>0.18</td>
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<tr>
<td>49.00</td>
<td>0.03</td>
<td>10,619</td>
</tr>
<tr>
<td>50.00</td>
<td>0.01</td>
<td>14,219</td>
</tr>
</tbody>
</table>

The spot price for ABC is $47.58. Estimate the premium for the at-the-money Call option in the following ways:

(a) Unweighted quadratic regression.

(b) Weighted quadratic regression.

(c) Polynomial interpolation.

(d) Spline interpolation.