Math 456: Introduction to Financial Mathematics

Homework Set 3

Due 23 October 2023

1. Suppose that $S(t,\omega)$, $0 \le t \le T$ is the price of a risky asset S, and that the riskless return over time $[0,T]$ is R. Model the future at time $t=T$ using $\Omega = \{\uparrow, \downarrow\}$ and assume that $S(T,\downarrow) < S(T,\uparrow)$.

   (a) Use the no-arbitrage Axiom 1 to conclude that
   $$S(T,\downarrow) < RS(0) < S(T,\uparrow).$$

   (b) Use the Fair Price Theorem 1.4 to prove the same inequalities.

2. In Exercise 1 above, model the future at time $t=T$ using the N-step binomial model $\Omega = \{\omega_0, \omega_1, \ldots, \omega_N\}$ and assume that
   $$S(T,\omega_k) = S(0)u^k d^{N-k},$$
   where $S(0) > 0$ is the spot price and $0 < d < u$ are the up factor and down factor, respectively, over one time step T/N.

   (a) Use the no-arbitrage Axiom 1 to conclude that
   $$d < R^{1/N} < u.$$

   (b) Use the Fair Price Theorem 1.4 to prove the same inequalities.

3. Suppose that a portfolio X contains risky stock S and riskless bond B in amounts h_0, h_1:
   $$X(t,\omega) = h_0 B(t,\omega) + h_1 S(t,\omega).$$
   Model the future at time $t=T$ using $\Omega = \{\uparrow, \downarrow\}$, assuming only that $S(T,\uparrow) \neq S(T,\downarrow)$ and that $B(T,\uparrow) = B(T,\downarrow) = R$. Compute h_0 and h_1 in terms of all the other quantities. (Hint: use Macsyma to derive Eq.3.1.)

4. In Exercise 3 above, suppose that X is a European-style Call option for S with expiry T and strike price K. Use the payoff formula $X(T) = [S(T)-K]^+$ in the equation for h_1 to prove that
   $$0 \le h_1 \le 1.$$
   Conclude that, in this model of the future, a European-style Call option for S is equivalent to a portfolio containing part of a share of S plus or minus some cash.

5. In Exercise 3 above, suppose that X is a European-style Put option for S with expiry T and strike price K. Use the payoff formula $X(T) = [K - S(T)]^+$ in the equation for h_1 to prove that

$$-1 \leq h_1 \leq 0.$$

Conclude that, in this model of the future, a European-style Put option for S is equivalent to a portfolio containing part of a share of S sold short plus or minus some cash.

6. Suppose that $C(0)$ and $P(0)$ are the premiums for European-style Call and Put options, respectively, on an asset S with the following parameters: expiry at $T = 1$ year, spot price $S(0) = 90$, strike price $K = 95$. Assume that the riskless annual percentage rate is $r = 0.02$, and the volatility for S is $\sigma = 0.15$, and that these will remain constant from now until expiry.

   (a) Use a LibreOffice Calc spreadsheet to implement the Cox-Ross-Rubinstein (CRR) model to compute $C(0)$ and $P(0)$ with $N = 10$ time steps, using the backward pricing formula in Eq.3.18. (Hint: compare output with `CRReurAD()` to check for bugs.)

   (b) Use the Octave function `CRReurAD()` with $N = 10$, $N = 100$, and $N = 1000$ time steps to compute $C(0)$ and $P(0)$.

   (c) Repeat part (b) with the Octave function `CRReur()` on p.88, again using $N = 10$, $N = 100$, and $N = 1000$ time steps to compute $C(0)$ and $P(0)$. Profile the time required to compute them, and compare the time and the output with that of `CRReurAD()`.

   (d) Compare the prices from parts (b) and (c). Is it justified to use $N = 1000$? Is $N = 10$ sufficiently accurate?

7. Compare the prices from parts (a) and (b) of previous Exercise 6 with the Black-Scholes prices computed using Eqs.2.25 and 2.26. Plot the logarithm of the differences against $\log N$ to estimate the rate of convergence. (Hint: Use the programs in Chapter 2, Section 2.4.)

8. Derive 3.32 on p.79:

$$q = \frac{1}{2} + \frac{r + \frac{\sigma^2}{2}}{2\sigma}\sqrt{\frac{T}{N}} + O\left(\frac{T}{N}\right).$$

9. Use the CRR approximation with $N = 4$ to compute the European-style Call option premiums at several hundred equally spaced spot prices $75 \leq S_0 \leq 115$, with expiry $T = 1$, strike $K = 95$, $r = 0.02$, and $\sigma = 0.15$.

   (a) Plot the values against S_0.

   (b) At what values of S_0 in that range does the graph appear to be nonsmooth?

   (c) Compute the points of nondifferentiability for S_0 in $[75, 115]$.

10. Compute the CRR option premiums and Greeks for European-style Call and Put options on a risky asset with the following parameters: spot price \$90, strike price \$95, expiry in 1 year, annual riskless rate 2%, and volatility 15%. Use $N = 100$ steps. Justify the method used.