Math 456: Introduction to Financial Mathematics

Homework Set 3
Due 23 October 2022

1. Suppose that \( S(t, \omega), 0 \leq t \leq T \) is the price of a risky asset \( S \), and that the riskless return over time \([0,T]\) is \( R \). Model the future at time \( t = T \) using \( \Omega = \{ \uparrow, \downarrow \} \) and assume that \( S(T, \downarrow) < S(T, \uparrow) \).
   
   (a) Use the no-arbitrage Axiom 1 to conclude that
   \[
   S(T, \downarrow) < RS(0) < S(T, \uparrow).
   \]
   
   (b) Use the Fair Price Theorem 1.4 to prove the same inequalities.

2. In Exercise 1 above, model the future at time \( t = T \) using the \( N \)-step binomial model \( \Omega = \{ \omega_0, \omega_1, \ldots, \omega_N \} \) and assume that
   \[
   S(T, \omega_k) = S(0)u^kd^{N-k},
   \]
   where \( S(0) > 0 \) is the spot price and \( 0 < d < u \) are the up factor and down factor, respectively, over one time step \( T/N \).
   
   (a) Use the no-arbitrage Axiom 1 to conclude that
   \[
   d < R^{1/N} < u.
   \]
   
   (b) Use the Fair Price Theorem 1.4 to prove the same inequalities.

3. Suppose that a portfolio \( X \) contains risky stock \( S \) and riskless bond \( B \) in amounts \( h_0, h_1 \):
   \[
   X(t, \omega) = h_0B(t, \omega) + h_1S(t, \omega).
   \]
   Model the future at time \( t = T \) using \( \Omega = \{ \uparrow, \downarrow \} \), assuming only that \( S(T, \uparrow) \neq S(T, \downarrow) \) and that \( B(T, \uparrow) = B(T, \downarrow) = R \). Compute \( h_0 \) and \( h_1 \) in terms of all the other quantities. (Hint: use Macsyma to derive Eq.3.1.)

4. In Exercise 3 above, suppose that \( X \) is a European-style Call option for \( S \) with expiry \( T \) and strike price \( K \). Use the payoff formula \( X(T) = [S(T) - K]^+ \) in the equation for \( h_1 \) to prove that
   \[
   0 \leq h_1 \leq 1.
   \]
   Conclude that, in this model of the future, a European-style Call option for \( S \) is equivalent to a portfolio containing part of a share of \( S \) plus or minus some cash.
5. In Exercise 3 above, suppose that $X$ is a European-style Put option for $S$ with expiry $T$ and strike price $K$. Use the payoff formula $X(T) = [K - S(T)]^+$ in the equation for $h_1$ to prove that

$$-1 \leq h_1 \leq 0.$$  

Conclude that, in this model of the future, a European-style Put option for $S$ is equivalent to a portfolio containing part of a share of $S$ sold short plus or minus some cash.

6. Suppose that $C(0)$ and $P(0)$ are the premiums for European-style Call and Put options, respectively, on an asset $S$ with the following parameters: expiry at $T = 1$ year, spot price $S(0) = 90$, strike price $K = 95$. Assume that the riskless annual percentage rate is $r = 0.02$, and the volatility for $S$ is $\sigma = 0.15$, and that these will remain constant from now until expiry.

(a) Use a LibreOffice Calc spreadsheet to implement the Cox-Ross-Rubinstein (CRR) model to compute $C(0)$ and $P(0)$ with $N = 10$ time steps, using the backward pricing formula in Eq.3.18. (Hint: compare output with CRReurAD() to check for bugs.)

(b) Use the Octave function CRReurAD() with $N = 10$, $N = 100$, and $N = 1000$ time steps to compute $C(0)$ and $P(0)$.

(c) Repeat part (b) with the Octave function CRReur() on p.88, again using $N = 10$, $N = 100$, and $N = 1000$ time steps to compute $C(0)$ and $P(0)$. Profile the time required to compute them, and compare the time and the output with that of CRReurAD().

(d) Compare the prices from parts (b) and (c). Is it justified to use $N = 1000$? Is $N = 10$ sufficiently accurate?

7. Compare the prices from parts (a) and (b) of previous Exercise 6 with the Black-Scholes prices computed using Eqs.2.25 and 2.26. Plot the logarithm of the differences against $\log(N)$ to estimate the rate of convergence. (Hint: Use the programs in Chapter 2, Section 2.4.)

8. Derive 3.32 on p.79:

$$q = \frac{1}{2} + \frac{r + \sigma^2}{2\sigma^2} \sqrt{\frac{T}{N}} + O\left(\frac{T}{N}\right).$$

9. Use the CRR approximation with $N = 4$ to compute the European-style Call option premiums at several hundred equally spaced spot prices $75 \leq S_0 \leq 115$, with expiry $T = 1$, strike $K = 95$, $r = 0.02$, and $\sigma = 0.15$.

(a) Plot the values against $S_0$.

(b) At what values of $S_0$ in that range does the graph appear to be nonsmooth?

(c) Compute the points of nondifferentiability for $S_0$ in $[75, 115]$.  

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10. Compute the CRR option premiums and Greeks for European-style Call and Put options on a risky asset with the following parameters: spot price $90, strike price $95, expiry in 1 year, annual riskless rate 2%, and volatility 15%. Use $N = 100$ steps. Justify the method used.