# Math 456: Introduction to Financial Mathematics

## Homework Set 5

1. Suppose that $S$ and $R$ are modeled with a recombining binomial tree of $N \geq 1$ levels. Prove that

$$G(N-1,j) \;=\; F(N-1,j) \;=\; S(N-1,j)R(N-1,j)$$

for all states $j$.

2. Suppose that $S$ and $R$ are modeled with a recombining binomial tree of $N = 2$ levels. Prove that if $(R(1,1) - R(1,0))(S(1,1) - S(1,0)) > 0$, then $G(0,0) > F(0,0)$.

3. Suppose that $S$ and $R$ are modeled with a recombining binomial tree of $N > 2$ levels. Prove that if

$$(R(N-1,j+1) - R(N-1,j))(S(N-1,j+1) - S(N-1,j)) > 0$$

for all $j = 0, 1, ..., N-2$, then $G(0,0) > F(0,0)$.

4. Fix $N = 4$ in the function `FwdFut()` defined on p.135.

(a) Find inputs $S$ and $R$ such that $F(0) > G(0)$, namely `F(1,1)>G(1,1)` in the output of `[F,G]=FwdFut(S,R,N)`.

(b) Find $S$ and $R$ such that $F(0) < G(0)$, namely `F(1,1)<G(1,1)` in the output of `[F,G]=FwdFut(S,R,N)`.

**Note:** To be valid, solutions $S$ and $R$ in parts (a) and (b) must be positive and must satisfy the no-arbitrage condition

$$S(n+1,j) < R(n,j)S(n,j) < S(n+1,j+1)$$

for all $0 \leq n < N$ and all $0 \leq j \leq n$.

5. Suppose that a commodities exchange wishes to broker Futures contracts on an asset $S$, expiring in 0.5 years while riskless annual interest rates are expected to remain constant at 0.02%. Under consideration are margin requirements of 20, 30, 50, 80, and 150% of $S_0$ for each contract.

(a) Compute the probability of a margin call, which will occur if and only if the margin balance falls below 0, for both Long and Short Futures contracts, with these five margin requirements. Use $N = 6$ time steps and volatilities $\sigma \in \{0.10, 0.15, 0.20, 0.25, 0.30, 0.40, 0.50, 0.70\}$. Tabulate the results and compare Long and Short margin requirements.

(b) Profile the computation with $N = 6$ and again with $N = 13$ to compare the run times. Include in the count all times above 1% of the total. Compare the run times to test the $O(2^N)$ order of complexity.

6. Suppose that $S$ is a dividend paying stock with a declared dividend amount $D$ at ex-dividend date $t^e$. Let $S(t^e-)$ denote the closing price of $S$ on the day before the ex-dividend date, and $S(t^e+)$ denote the opening price of $S$ on the ex-dividend date. Show that

$$S(t^e+) = S(t^e-) - D$$

by constructing an arbitrage otherwise.

7. Prove Eq.6.9 from Eq.6.8.

8. Use the data in Table 6.1 (p.146) and Table 6.2 (p.154) to compute the present value sequence of BAC dividends from December 2014 through March 2021, namely time indices 0–25. Disregard any dividends outside of that time period.

   (a) Find the values quarterly.

   (b) Find the values monthly.

   (c) Plot the two dividend present value sequences, and the dividends themselves, on the same graph.

9. For the experiments below, use `CRRDaeC()` with the parameters $T = 1$, $S_0 = 100$, $K = 101$, $r = 0.02$, $v = 0.20$, and $N = 12$.

   (a) Find a nonzero dividend sequence for which early Call exercise is sometimes optimal.

   (b) Find another nonzero dividend sequence for which early Call exercise is never optimal.

10. Implement CRR pricing for American and European Put options on an asset with dividends, using decomposition into risky ex-dividend and riskless cash flow portions. (Hint: Make a few changes to `CRRDaeC()`.)

    For the experiments below, set $T = 1$, $S_0 = 100$, $K = 101$, $r = 0.02$, $v = 0.20$, and $N = 12$.

    (a) Find a nonzero dividend sequence for which early Put exercise is sometimes optimal.

    (b) Find a nonzero dividend sequence for which early Put exercise is never optimal.

11. Suppose that government bonds are available with maturities of 1, 2, 3, and 4 years, with annual coupons for 0.5, 0.6, 0.8, and 1.1% of face value, respectively. Their spot prices are respectively 0.9994, 0.9992, 0.9989, and 0.9985 times face value.

(a) Compute the zero-coupon bond discounts $Z(0,1)$, $Z(0,2)$, $Z(0,3)$, and $Z(0,4)$ implied by these inputs.

(b) Plot the yield curve implied by $\{Z(0,T) : T = 1,2,3,4\}$.

12. Suppose that a company offers coupon bonds with maturities of 1, 2, 3, and 4 years, with semiannual coupons at 0.3, 0.4, 0.5, and 0.6% of face value, respectively. Their spot prices are respectively 0.9994, 0.9992, 0.9989, and 0.9985 times face value.

    (a) Compute the zero-coupon bond discounts $\hat{Z}(0,1)$, $\hat{Z}(0,2)$, $\hat{Z}(0,3)$, and $\hat{Z}(0,4)$ implied by these inputs.

    (b) Plot the yield curve implied by $\{\hat{Z}(0,T) : 0 < T \leq 4\}$.

13. Compute the price at issuance of a US Treasury Note with the following parameters:

    - Maturity in 7 years.
    - Semiannual coupon at 1.500% annual interest.
    - Face value \$1000.
    - Yield to maturity 1.414%.

    Use both $e^x$ and its approximations $1 + x$ and $1 - x$ in the present value calculation and compare the results.

14. Compute the expected monthly riskless rates and returns over 6 months for two currencies using tabulated benchmarks:

    (a) Use the data in Table 6.5 for 11 March 2022 to compute the Australian dollar values.

    (b) Use the LIBOR data in Table 6.7 for 11 March 2022 to compute the US dollar values. Justify your interpolation method.