Math 456: Introduction to Financial Mathematics

Homework Set 6

1. Suppose that C and P are European-style Call and Put options, respectively, at strike price K and expiry T, for a risky underlying asset S with spot price S_0. Show that $0 < C(0) \leq S_0$ and $0 < P(0) \leq K$. (Hint: construct an arbitrage otherwise.)

2. Show that the Eq.7.11 and Eq.7.12 probabilities produce the Arrow-Debreu spot prices $\lambda(n, j)$ in Eq.7.10 using Jamshidian's forward induction, Eq.3.21 on p.69.

3. Compute implied volatility for the data in Table 7.1 using both Black-Scholes and CRR with $N = 20$. Tabulate and compare the results.

4. The table below gives part of the options chain for American-style Calls on Bank of America common stock (BAC) as of closing on March 17, 2022, when the spot price was \$43.03:

Strike price K	T= 1 d (3/18)	8 d (3/25)	15 d (4/01)	21 d (4/08)	27 d (4/14)
42.00	1.10	1.44	1.76	1.96	2.18
43.00	0.34	0.86	1.10	1.33	1.58
44.00	0.06	0.44	0.69	0.88	1.06
45.00	0.02	0.18	0.35	0.53	0.71
46.00	0.01	0.07	0.16	0.31	0.44
47.00	0.01	0.02	0.09	0.17	0.27

   Also, the US T-bill rates for various maturities were

Date	4 wk	8 wk	13 wk	26 wk	52 wk
03/14/2022	0.22	0.30	0.45	0.84	1.20
03/15/2022	0.21	0.29	0.46	0.84	1.19
03/16/2022	0.23	0.28	0.43	0.84	1.26
03/17/2022	0.20	0.30	0.40	0.79	1.20

   Use this data to compute and plot the volatility surface for BAC.

5. Suppose that a share of XYZ has a spot price of \$47.12, that riskless interest rates for the next month are expected to be a constant 0.66% APR, and that the premiums for European-style Call options expiring in 4 weeks ($T = 4/52$) are as follows:

Strike price:	45.00	46.00	47.00	48.00	49.00
Call premium:	3.52	2.78	2.10	1.44	1.37

(a) Construct an implied binomial tree for these inputs using Rubinstein's 1-2-3 algorithm. Display it along with the implied risk neutral up probabilities.

(b) Plot the three weight functions $w_1(x) = \sqrt{x}$, $w_2(x) = x^2$, and $w_3(x) = (1 - \cos(\pi x))/2$, for $0 \leq x \leq 1$, on the same graph.

(c) Apply Rubinstein's 1-2-3 algorithm with Jackwerth's generalization to the data, using weights w_1, w_2, w_3 from part (b). Compare S, p, and Q for the three weights.

6. Prove that any subspace $V \subset \mathbf{R}^n$ is a closed convex cone.

7. Prove that the closed orthant $K \in \mathbf{R}^n$ of vectors with nonnegative coordinates is a closed convex cone.

8. Prove that the pointless orthant $K \setminus \mathbf{0}$ is a convex cone but is neither open nor closed.

9. Prove that K^o is an open convex cone.

10. Prove that the intersection of any collection of convex sets is convex.

11. Prove Theorem 8.16 on p.209:

    (a) $K' = K$, that is, the nonnegative orthant is a self-dual cone.

    (b) $(K^o)' = K$ and $(K^o)^* = K \setminus \mathbf{0}$.

    (c) $(K \setminus \mathbf{0})' = K$ and $(K \setminus \mathbf{0})^* = K^o$.

    (d) $((K^o)^*)^* = K^o$, that is, the open positive orthant is its own strict double dual cone.

12. Prove Eq.8.6:
$$AK = \sum_{i=1}^{n} \bar{V}_i; \qquad AK^o = \sum_{i=1}^{n} V_i,$$
where $A \in \mathbf{R}^{m \times n}$, and K, K^o are the orthants of Definition 6.

13. Prove Corollary 8.18 on p.209: The set S of strictly profitable portfolios is a strict dual cone: $S = (AK^o)^*$

14. Suppose $S \subset \mathbf{R}^n$ is any set. Prove the following:

    (a) $S^\perp$ is a subspace.

    (b) $S^* \subset S'$ and thus $S^* \cap S' = S^*$.

    (c) $S^\perp \subset S'$ and thus $S^\perp \cap S' = S^\perp$.

    (d) $S^\perp \cap S^* = \emptyset$.

(e) $S^{\perp}$, S', and S^* are all convex cones.

(f) If $\mathbf{0} \in S$, then $S^* = \emptyset$. Thus if S is a subspace, then $S^* = \emptyset$.

15. Suppose that $n > 2$ and market model $A, \mathbf{q}$ has

$$A = \begin{pmatrix} R & \cdots & R \\ a_1 & \cdots & a_n \end{pmatrix},$$

where $R > 1$ is the riskless return and $\mathbf{a} = (a_1, \ldots, a_n)$ is a nonconstant payoff vector for the sole risky asset.

(a) Find necessary and sufficient conditions on $\mathbf{q}$ such that $A, \mathbf{q}$ is arbitrage-free. (Hint: use the Fundamental Theorem.)

(b) Exhibit a derivative payoff $\mathbf{d}$ for which no exact hedge exists. (This shows that A is not a complete market.)

(c) Exhibit a derivative $\mathbf{d}$ for which an exact hedge does exist.

16. Suppose that a market model has five states, a riskless asset returning $R = 1.02$, and two risky assets a, b with spot prices $a_0 = 20$ and $b_0 = 12$ and payoffs $\mathbf{a} = (10, 15, 20, 25, 30)$ and $\mathbf{b} = (17, 15, 12, 10, 7)$, respectively.

(a) Prove that the model is arbitrage-free.

(b) Find the no-arbitrage bid-ask interval for a European-style Call option on a with strike price 20.

(c) Find the no-arbitrage bid-ask interval for a European-style Put option on b with strike price 13.