1. Suppose that $C$ and $P$ are European-style Call and Put options, respectively, at strike price $K$ and expiry $T$, for a risky underlying asset $S$ with spot price $S_0$. Show that $0 < C(0) \leq S_0$ and $0 < P(0) \leq K$. (Hint: construct an arbitrage otherwise.)

**Solution:** First note that $C(0) > 0$ and $P(0) > 0$ are consequences of Theorem 1.1.

Next, suppose that $C(0) > S_0$. Sell one $C$ and buy one $S$ at time $t = 0$, keeping the surplus $C(0) - S_0 > 0$. At expiry $T$, if the buyer exercises $C$, collect $K > 0$ in exchange for $S$. Otherwise, sell $S$ for $S(T) > 0$.

Finally, suppose that $P(0) > K$. Sell one $P$ for $P(0)$ at $t = 0$ and keep the proceeds. At expiry $t = T$, if the buyer exercises $P$, receive that $S$ at strike price $K$, leaving a surplus $P(0) - K > 0$ and sell it for $S(T) > 0$, netting additional profit. Otherwise, keep the original $P(0)$ with no further obligations.

In all cases there is a positive payoff with no initial investment. Conclude by the no-arbitrage axiom that $0 < C(0) \leq S_0$ and $0 < P(0) \leq K$. □

2. Show that the Eq.7.11 and Eq.7.12 probabilities produce the Arrow-Debreu spot prices $\lambda(n, j)$ in Eq.7.10 using Jamshidian’s forward induction, Eq.3.21 on p.69.

**Solution:** This may be proved by induction on $n$. For $n = 0$, the only Arrow-Debreu spot price is $\lambda(0, 0) = 1$, so it agrees with the value $Q(0, 0)/R^0 = 1$ from Jackwerth’s construction.

Now suppose that the values $\{\lambda(n-1, j) : j = 0, 1, \ldots, n-1\}$ produced by Jamshidian’s induction agree with Jackwerth’s values and that $p$ and $1 - p$ are given by Eq.7.11 and Eq.7.12, respectively. (As usual, take $\lambda(n, j) = 0$ if $j < 0$ or $j > n$.) Compute $\lambda(n, j)$ for $j = 0, 1, \ldots, n$ by substituting the
expressions from Jackwerth’s construction into Eq.3.21:

\[ \lambda(n,j) = 1 - p(n-1,j) \lambda(n-1,j) + \frac{p(n-1,j-1)}{R} \lambda(n-1,j-1) \]

\[ = \frac{1}{R} \left[ 1 - w \left( \frac{j}{n} \right) \right] Q(n,j) + \frac{1}{R} \left[ \frac{Q(n,j)}{Q(n-1,j-1)} \right] Q(n-1,j-1) \]

\[ = \left[ \left( 1 - w \left( \frac{j}{n} \right) \right) + w \left( \frac{j}{n} \right) \right] \frac{Q(n,j)}{R^n} = \frac{Q(n,j)}{R^n}. \]

Thus Jamshidian’s forward induction with Jackwerth’s probabilities produces Jackwerth’s Arrow-Debreu spot prices.

3. Compute implied volatility for the data in Table 7.1 using both Black-Scholes and CRR with \( N = 20 \). Tabulate and compare the results.

**Solution:** Use the Octave code that produced Figure 7.1, with the following modifications:

```octave
Ks=40:46; Ts=[3,9,21,35,63]; % strikes and days to expiry
C=[4.14 4.35 4.85 4.60 5.15; 3.20 3.42 3.50 4.00 4.35; 2.24 2.72 2.89 3.15 3.65; 1.30 1.51 1.96 2.44 2.87; 0.63 0.86 1.34 1.71 2.39; 0.23 0.45 0.84 1.24 1.76; 0.07 0.22 0.52 0.88 1.39]; % Call premiums on 2021-12-14
S0=44.13; r=0.05; % spot price and riskless APR on 2021-12-14
VimpBS=zeros(length(Ks),length(Ts)); % BS implied volatilities
VimpCRR=zeros(length(Ks),length(Ts)); % CRR implied volatilities
minv=0.01; maxv=0.99; tol=0.00001; % bisection parameters
for col=1:length(Ts)
    T=Ts(col)/365; % time to expiry in years
    for row=1:length(Ks)
        f=@(v) BS(T,S0,Ks(row),r/100,v); % Black-Scholes Call
        VimpBS(row,col)=bisection(f,C(row,col),minv,maxv,tol);
        g=@(v) CRReur(T,S0,Ks(row),r/100,v,20)(1,1); % CRR Call
        VimpCRR(row,col)=bisection(g,C(row,col),minv,maxv,tol);
    end
end
```

The output is in Table 1:

**Remark.** As expected, the two methods produce nearly identical values. To further check the results and the code, compare the Black-Scholes (BS) values with the \( \sigma \) columns of Table 7.1.
Table 1: Implied volatilities by CRR ($N = 20$) and Black-Scholes methods, from December 14, 2021 closing prices for BAC American-style Call options at $r = 0.05\%$ and $S_0 = $44.13.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Dec17 CRR</th>
<th>Dec23 CRR</th>
<th>Jan7 CRR</th>
<th>Jan21 CRR</th>
<th>Feb18 CRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.50</td>
<td>0.54</td>
<td>0.54</td>
<td>0.36</td>
<td>0.37</td>
</tr>
<tr>
<td>41</td>
<td>0.56</td>
<td>0.49</td>
<td>0.35</td>
<td>0.39</td>
<td>0.35</td>
</tr>
<tr>
<td>42</td>
<td>0.47</td>
<td>0.52</td>
<td>0.39</td>
<td>0.36</td>
<td>0.34</td>
</tr>
<tr>
<td>43</td>
<td>0.36</td>
<td>0.30</td>
<td>0.31</td>
<td>0.33</td>
<td>0.31</td>
</tr>
<tr>
<td>44</td>
<td>0.35</td>
<td>0.29</td>
<td>0.30</td>
<td>0.30</td>
<td>0.32</td>
</tr>
<tr>
<td>45</td>
<td>0.34</td>
<td>0.29</td>
<td>0.28</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>46</td>
<td>0.36</td>
<td>0.30</td>
<td>0.28</td>
<td>0.30</td>
<td>0.29</td>
</tr>
</tbody>
</table>

4. The table below gives part of the options chain for American-style Calls on Bank of America common stock (BAC) as of closing on March 17, 2022, when the spot price was $43.03:

<table>
<thead>
<tr>
<th>Strike</th>
<th>T=</th>
<th>1 d</th>
<th>8 d</th>
<th>15 d</th>
<th>21 d</th>
<th>27 d</th>
</tr>
</thead>
<tbody>
<tr>
<td>42.00</td>
<td>1.10</td>
<td>1.44</td>
<td>1.76</td>
<td>1.96</td>
<td>2.18</td>
<td></td>
</tr>
<tr>
<td>43.00</td>
<td>0.34</td>
<td>0.86</td>
<td>1.10</td>
<td>1.33</td>
<td>1.58</td>
<td></td>
</tr>
<tr>
<td>44.00</td>
<td>0.06</td>
<td>0.44</td>
<td>0.69</td>
<td>0.88</td>
<td>1.06</td>
<td></td>
</tr>
<tr>
<td>45.00</td>
<td>0.02</td>
<td>0.18</td>
<td>0.35</td>
<td>0.53</td>
<td>0.71</td>
<td></td>
</tr>
<tr>
<td>46.00</td>
<td>0.01</td>
<td>0.07</td>
<td>0.16</td>
<td>0.31</td>
<td>0.44</td>
<td></td>
</tr>
<tr>
<td>47.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.09</td>
<td>0.17</td>
<td>0.27</td>
<td></td>
</tr>
</tbody>
</table>

Also, the US T-bill rates for various maturities were

<table>
<thead>
<tr>
<th>Date</th>
<th>4 wk</th>
<th>8 wk</th>
<th>13 wk</th>
<th>26 wk</th>
<th>52 wk</th>
</tr>
</thead>
<tbody>
<tr>
<td>03/14/2022</td>
<td>0.22</td>
<td>0.30</td>
<td>0.45</td>
<td>0.84</td>
<td>1.20</td>
</tr>
<tr>
<td>03/15/2022</td>
<td>0.21</td>
<td>0.29</td>
<td>0.46</td>
<td>0.84</td>
<td>1.19</td>
</tr>
<tr>
<td>03/16/2022</td>
<td>0.23</td>
<td>0.28</td>
<td>0.43</td>
<td>0.84</td>
<td>1.26</td>
</tr>
<tr>
<td>03/17/2022</td>
<td>0.20</td>
<td>0.30</td>
<td>0.40</td>
<td>0.79</td>
<td>1.20</td>
</tr>
</tbody>
</table>

Use this data to compute and plot the volatility surface for BAC.

**Solution:** Use the commands that produced Figure 7.1, but with the data for this problem. Set the riskless rate for all calculations to be the 4-week rate averaged over the 4 days sampled, which is 0.215% APR.

```
Ks=42:47; Ts=[1,8,15,21,27]; % strikes and days to expiry
C=[1.10 1.44 1.76 1.96 2.18 ;... % Call premiums on 2022-03-17
0.34 0.86 1.10 1.33 1.58 ;...%
0.06 0.44 0.69 0.88 1.06 ;...%
0.02 0.18 0.35 0.53 0.71 ;...%
0.01 0.07 0.16 0.31 0.44 ;...%
0.01 0.02 0.09 0.17 0.27 ]; % Call premiums on 2022-03-17
```

3
S0=43.03; r=0.215; % spot price and riskless APR
Vimp=zeros(length(Ks),length(Ts)); % implied volatilities
minv=0.01; maxv=0.99; tol=0.00001; % bisection parameters
for col=1:length(Ts)
    T=Ts(col)/365; % time to expiry in years
    for row=1:length(Ks)
        f=@(v) BS(T,S0,Ks(row),r/100,v); % Black-Scholes Call
        Vimp(row,col)=bisection(f,C(row,col),minv,maxv,tol);
    end
end
mesh(Ts,Ks,Vimp); % note the transposed order (T,K,V(K,T))
title("Implied Volatility Surface");
xlabel("T (days)"); ylabel("K");

Figure 1: Graph from Exercise 4.

The results may be seen in Figure 1.

5. Suppose that a share of XYZ has a spot price of $47.12, that riskless interest rates for the next month are expected to be a constant 0.66% APR, and that the premiums for European-style Call options expiring in 4 weeks ($T = 4/52$) are as follows:

<table>
<thead>
<tr>
<th>Strike price</th>
<th>45.00</th>
<th>46.00</th>
<th>47.00</th>
<th>48.00</th>
<th>49.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call premium</td>
<td>3.52</td>
<td>2.78</td>
<td>2.10</td>
<td>1.44</td>
<td>1.37</td>
</tr>
</tbody>
</table>

(a) Construct an implied binomial tree for these inputs using Rubinstein’s 1-2-3 algorithm. Display it along with the implied risk neutral up probabilities.
(b) Plot the three weight functions \( w_1(x) = \sqrt{x}, \ w_2(x) = x^2, \) and \( w_3(x) = \frac{1 - \cos(\pi x)}{2} \), for \( 0 \leq x \leq 1 \), on the same graph.

(c) Apply Rubinstein’s 1-2-3 algorithm with Jackwerth’s generalization to the data, using weights \( w_1, w_2, w_3 \) from part (b). Compare \( S, p, \) and \( Q \) for the three weights.

Solution:

(a) Apply Rubinstein’s original 1-2-3 algorithm using IBT123J.m with the following Octave commands:

\[
\begin{align*}
\text{Ks} & = [45, 46, 47, 48, 49]; \\
\text{Cs} & = [3.52, 2.78, 2.10, 1.44, 1.37]; \\
\text{S0} & = 47.12; \ x = 0.66/100; \ \rho = \exp(r); \ w = @(x)x; \\
[S, Q, \text{up}, \text{down}, \text{pu}, \text{N1}] & = \text{IBT123J}(\text{S0}, \text{Ks}, \text{Cs}, \rho, w); \ S, \text{pu}
\end{align*}
\]

That produces the following output, edited for space savings:

\[
\begin{align*}
S & = \% \text{ implied binomial tree} \quad \text{pu} = \% \text{ risk neutral up probs.} \\
47.12 & \\
43.48 & 50.36 \quad 0.3710 \quad 0.6853 \\
41.14 & 47.63 \quad 51.73 \quad 0.0639 \quad 0.8916 \quad 0.5905 \\
40.88 & 46.11 \quad 47.90 \quad 54.53 \quad 0.0559 \quad 0.1818 \quad 0.9779 \quad 0.3218 \\
40.64 & 46.00 \quad 47.00 \quad 48.00 \quad 68.57
\end{align*}
\]

(b) The three weight functions all differ from \( w(x) = x \): \( w_1(x) = \sqrt{x} \) is concave, \( w_2(x) = x^2 \) is convex, and \( w_3(x) = \frac{1 - \cos(\pi x)}{2} \) has a unique inflection point. They may be plotted as in Figure 2 using the following Octave/MATLAB commands:

\[
\begin{align*}
w1 & = @(x)\sqrt{x}; \ w2 = @(x)x.^2; \ w3 = @(x)(1-\cos(\pi x))/2; \\
t & = 0:0.01:1; \ \text{plot}(t, w1(t), "r--", t, w2(t), "b..", t, w3(t), "k-"); \\
\text{legend}("w1", "w2", "w3", "location", "southeast"); \\
\text{title}("Weight Functions for Jackwerth's Generalization")
\end{align*}
\]

(c) Run three experiments as follows, reusing previously assigned variables from parts (a) and (b):

\[
\begin{align*}
[S1, Q1, \text{up}, \text{down}, \text{pu1}, \text{N1}] & = \text{IBT123J}(\text{S0}, \text{Ks}, \text{Cs}, \rho, w1); \ S1, Q1, \text{pu1} \\
[S2, Q2, \text{up}, \text{down}, \text{pu2}, \text{N1}] & = \text{IBT123J}(\text{S0}, \text{Ks}, \text{Cs}, \rho, w2); \ S2, Q2, \text{pu2} \\
[S3, Q3, \text{up}, \text{down}, \text{pu3}, \text{N1}] & = \text{IBT123J}(\text{S0}, \text{Ks}, \text{Cs}, \rho, w3); \ S3, Q3, \text{pu3}
\end{align*}
\]

The results, side-by-side, are:

\[
\begin{align*}
\text{S1} & = \quad \text{S2} = \quad \text{S3} = \\
47.12 & \\
44.57 & 51.57 \quad 41.74 \quad 49.74 \quad 43.82 \quad 50.53 \\
41.50 & 47.76 \quad 53.72 \quad 40.73 \quad 47.31 \quad 50.58 \quad 40.98 \quad 47.63 \quad 52.87 \\
41.14 & 46.24 \quad 47.91 \quad 57.57 \quad 40.65 \quad 46.01 \quad 47.88 \quad 52.30 \quad 40.76 \quad 46.09 \quad 47.90 \quad 57.11 \\
\% \ \text{each last S row is 40.64, 46.00, 47.00, 48.00, 68.57} \\
Q1 & = \quad Q2 = \quad Q3 = \\
1 & 1 & 1
\end{align*}
\]
Weight Functions for Jackwerth’s Generalization

Figure 2: Weight functions for Jackwerth’s generalization in Rubinstein’s 1-2-3 algorithm: $w_1(x) = \sqrt{x}$, $w_2(x) = x^2$, and $w_3(x) = (1 - \cos(\pi x))/2$.

$$
\begin{array}{ccc}
.6246 & .3754 & \cdot .3182 & .6818 & .4963 & .5037 \\
.3110 & .4436 & .2455 & .2657 & .2100 & .5243 & .2793 & .4340 & .2867 \\
.2853 & .0444 & .5202 & .1500 & .2589 & .0617 & .3492 & .3303 & .2639 & .0616 & .5170 & .1574 \\
\% \text{...each last Q row is } & 0.2551 & 0.0604 & 0.0201 & 0.5939 & 0.0705 \\
\text{pu1} = & \text{pu2} = & \text{pu3} = \\
.3754 & .6818 & .5037 \\
.6021 & .6539 & .1650 & .7690 & .4372 & .5692 \\
.0825 & .9577 & .6111 & .0258 & .7390 & .6300 & .0551 & .8935 & .5492 \\
\end{array}
$$

Note that there is one less row for the up probabilities.

6. Prove that any subspace $V \subset \mathbb{R}^n$ is a closed convex cone.

**Solution:** Check the needed properties:

**Cone:** $v \in V \implies \lambda v \in V$ for any $\lambda > 0$, since any multiple of a vector is still in the subspace.

**Convex:** For any $x, y \in V$ and any $\lambda \in [0,1]$, both $\lambda x \in V$ and $(1 - \lambda)y \in V$ just as above. Then $\lambda x + (1 - \lambda)y \in V$ because sums of vectors in $V$ remain in $V$. 

6
7. Prove that the closed orthant \( K \subset \mathbb{R}^n \) of vectors with nonnegative coordinates is a closed convex cone.

**Solution:** Check the needed properties:

- **Cone:** For any \( k = (k_1, \ldots, k_n) \in K \iff (\forall i) k_i \geq 0 \), so for any \( \lambda > 0 \), \( \lambda k = (\lambda k_1, \ldots, \lambda k_n) \in K \) because \((\forall i) \lambda k_i \geq 0\).

- **Convex:** For any \( x, y \in K \) and any \( \lambda \in [0,1] \), the coordinates of \( \lambda x + (1-\lambda)y \) will be
  \[
  \lambda x_i + (1-\lambda)y_i \geq 0, \quad i = 1, \ldots, n,
  \]
  since \( x_i, y_i, \lambda \), and \( 1-\lambda \) are all nonnegative. Hence \( \lambda x + (1-\lambda)y \in K \).

- **Closed:** It suffices to prove that the complement of \( K \) is open. But \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus K \iff (\exists i) x_i < 0 \). Suppose without loss of generality that \( x_1 < 0 \). Let \( \epsilon = |x_1|/2 \). Then for every \( y = (y_1, \ldots, y_n) \in B_r(x) \), it must be that \( |y_1 - x_1| < \epsilon = |x_1|/2 \), so \( y_1 < 0 \) as well. Conclude that \( B_r(x) \subset \mathbb{R}^n \setminus K \), so \( \mathbb{R}^n \setminus K \) is an open set, so \( K \) is closed. \( \square \)

8. Prove that the pointless orthant \( K \setminus 0 \) is a convex cone but is neither open nor closed.

**Solution:** Check the needed properties:

- **Cone:** \( k = (k_1, \ldots, k_n) \) belongs to \( K \setminus 0 \) if and only if all coordinates \( k_i \) are nonnegative and at least one of them is positive. This property is preserved by multiplication by \( \lambda > 0 \).

- **Convex:** For any \( x, y \in K \setminus 0 \) and any \( \lambda \in [0,1] \), the coordinates of \( \lambda x + (1-\lambda)y \) will be
  \[
  \lambda x_i + (1-\lambda)y_i \geq 0, \quad i = 1, \ldots, n,
  \]
  since \( x_i, y_i, \lambda \), and \( 1-\lambda \) are all nonnegative. Hence \( \lambda x + (1-\lambda)y \in K \). It remains to show that some coordinate is positive, which can be done by checking two cases for \( \lambda \in [0,1] \). By hypothesis, there is some \( i \) such that \( x_i > 0 \) and some \( j \) such \( y_j > 0 \).

  - If \( \lambda > 0 \), then \( \lambda x_i + (1-\lambda)y_i > 0 \).
  - Else \( \lambda = 0 \), so \( 1-\lambda = 1 > 0 \), so \( \lambda x_j + (1-\lambda)y_j = y_j > 0 \).

Conclude that \( \lambda x + (1-\lambda)y \in K \setminus 0 \).

**Not open:** Every open ball centered at the point \( (1,0,\ldots,0) \in K \setminus 0 \) contains points with some negative coordinates which are therefore not in \( K \setminus 0 \). Hence \( K \setminus 0 \) is not open.
Not closed: Every open ball \( B_r(0) \) centered at the point 0 in the complement of \( K \setminus 0 \) contains points with all positive coordinates which are therefore in \( K \setminus 0 \). Hence the complement of \( K \setminus 0 \) is not open. Hence \( K \setminus 0 \) is not closed.

9. Prove that \( K^o \) is an open convex cone.

Solution: Check the needed properties:

Cone: \( k = (k_1, \ldots, k_n) \) belongs to \( K^o \) if and only if all coordinates \( k_i \) are positive. This property is preserved by multiplication by \( \lambda > 0 \).

Convex: For any \( x, y \in K^o \) and any \( \lambda \in [0, 1] \), the coordinates of \( \lambda x + (1 - \lambda)y \) will be

\[
\lambda x_i + (1 - \lambda)y_i > 0, \quad i = 1, \ldots, n,
\]

since at least one of \( \lambda x_i \) or \( (1 - \lambda)y_i \) must be positive for every \( i \). Hence \( \lambda x + (1 - \lambda)y \in K^o \).

Open: Choose any \( x = (x_1, \ldots, x_n) \in K^o \). Let

\[
\epsilon = \min\{x_i/2 : i = 1, \ldots, n\}.
\]

Then \( \epsilon > 0 \) since \( x_i > 0 \) for every \( i \). But then every point \( y \in B_r(x) \) has coordinates satisfying

\[
|y_i - x_i| < \epsilon, \quad \implies \quad y_i > x_i - \epsilon > 0.
\]

Thus \( y \in K^o \), so \( B_r(x) \subset K^o \), so \( K^o \) is open.

10. Prove that the intersection of any collection of convex sets is convex.

Solution: Suppose that \( \{C_\alpha : \alpha \in I\} \) is an arbitrary collection of convex sets. Let \( S = \bigcap_{\alpha \in I} \) be the intersection of all of them.

If \( S = \emptyset \), then \( S \) is convex as there is nothing to check.

Otherwise, suppose \( x, y \in S \), fix \( t \in [0, 1] \), and let \( z = tx + (1 - t)x \). For every \( \alpha \in I \), \( x, y \in C_\alpha \) implies \( z \in C_\alpha \), since \( C_\alpha \) is convex. But then \( z \in S = \bigcap_\alpha C_\alpha \). Conclude that \( S \) is convex.

11. Prove Theorem 8.16 on p.209:

(a) \( K' = K \), that is, the nonnegative orthant is a self-dual cone.

(b) \( (K^o)^* = K \) and \( (K^o)* = K \setminus 0 \).

(c) \( (K \setminus 0)^* = K \) and \( (K \setminus 0)* = K^o \).

(d) \( ((K^o)*)^* = K^o \), that is, the open positive orthant is its own strict double dual cone.
**Solution:** (a) Choose any $\mathbf{x} = (x_1, \ldots, x_n) \in K$. Take $\mathbf{k} = (1, 0, \ldots, 0) \in K$ to compute $x_1 = \mathbf{x}^T \mathbf{k} \geq 0$. Corresponding arguments show that $x_i \geq 0$ for every $i = 1, 2, \ldots, n$. Thus $K' \subseteq K$.

Conversely, if $\mathbf{x} \in K$, then for every $\mathbf{k} \in K$ compute

$$x^T \mathbf{k} = x_1 k_1 + \cdots + x_n k_n \geq 0,$$

since all terms are nonnegative. Thus $K \subseteq K'$. Conclude that $K = K'$.

(b) To find the dual, suppose $\mathbf{x} = (x_1, \ldots, x_n) \in (K^o)'$. Take $\mathbf{k} = (1, \epsilon, \ldots, \epsilon) \in K^o$ to compute $x_1 + \epsilon(x_2 + \cdots x_n) = \mathbf{x}^T \mathbf{k} \geq 0$. If $x_1 < 0$, then for sufficiently small $\epsilon > 0$ this inequality will be violated. Thus it must be that $x_1 \geq 0$. Corresponding arguments show that $x_i \geq 0$ for every $i = 1, 2, \ldots, n$. Thus $(K^o)' \subseteq K$.

Conversely, if $\mathbf{x} \in K$, then for every $\mathbf{k} \in K^o$ compute

$$x^T \mathbf{k} = x_1 k_1 + \cdots + x_n k_n \geq 0,$$

since all factors and summands are nonnegative. Thus $K \subseteq (K^o)'$. Conclude that $K = (K^o)'$.

To find the strict dual, suppose $\mathbf{x} = (x_1, \ldots, x_n) \in (K^o)^*$. Take $\mathbf{k} = (1, \epsilon, \ldots, \epsilon) \in K^o$ to compute $x_1 + \epsilon(x_2 + \cdots x_n) = \mathbf{x}^T \mathbf{k} > 0$. If $x_1 < 0$, then, as before, for sufficiently small $\epsilon > 0$ this inequality will be violated. Thus $x_1 \geq 0$. Corresponding arguments show that $x_i \geq 0$ for every $i = 1, 2, \ldots, n$. But also, if $\mathbf{x} = 0$, then $\mathbf{x}^T \mathbf{k} = 0$ so the inequality will be violated. Thus $(K^o)^* \subseteq K \setminus 0$.

Conversely, if $\mathbf{x} \in K \setminus 0$, then for every $\mathbf{k} \in K^o$ compute

$$x^T \mathbf{k} = x_1 k_1 + \cdots + x_n k_n > 0,$$

since all factors and summands are nonnegative and at least one of them must be positive. Thus $K \setminus 0 \subseteq (K^o)^*$. Conclude that $K \setminus 0 = (K^o)^*$.

**Remark.** Parts (a) and (b) show that $A' = B'$ does not imply $A = B$.

(c) First, to find the dual, suppose $\mathbf{x} = (x_1, \ldots, x_n) \in (K \setminus 0)'$. Take $\mathbf{k} = (1, 0, \ldots, 0) \in K \setminus 0$ to compute $x_1 = \mathbf{x}^T \mathbf{k} \geq 0$. Corresponding arguments show that $x_i \geq 0$ for every $i = 1, 2, \ldots, n$. Thus $(K \setminus 0)' \subseteq K$.

Conversely, if $\mathbf{x} \in K$, then for every $\mathbf{k} \in K \setminus 0$ compute

$$x^T \mathbf{k} = x_1 k_1 + \cdots + x_n k_n \geq 0,$$

since all factors and summands are nonnegative. Thus $K \subseteq (K \setminus 0)'$. Conclude that $K = (K \setminus 0)'$.

Second, to find the strict dual, suppose $\mathbf{x} = (x_1, \ldots, x_n) \in (K \setminus 0)^*$. Take $\mathbf{k} = (1, 0, \ldots, 0) \in K \setminus 0$ to compute $x_1 = \mathbf{x}^T \mathbf{k} > 0$. Similarly, compute $x_i > 0$ for every $i = 1, 2, \ldots, n$. Thus $(K \setminus 0)^* \subseteq K^o$.
Conversely, if \( x \in K^o \), then for every \( k \in (K \setminus 0)^* \) compute
\[
x^T k = x_1 k_1 + \cdots + x_n k_n > 0,
\]
since all factors and summands are nonnegative and at least one of them must be positive. Thus \( K \setminus 0 \subset (K^o)^* \). Conclude that \( K \setminus 0 = (K^o)^* \).

(d) Observe that if \( x \in K^o \), then \( x^T k > 0 \) for every \( k \in (K^o)^* \). Thus \( K^o \subset ((K^o)^*)^* \).

Conversely, if \( x \in ((K^o)^*)^* \), then choosing \( k = (1, 0, \ldots, 0) \in K \setminus 0 = (K^o)^* \), as shown in part (b), gives \( x_1 = x^T k > 0 \). Similarly, \( x_i > 0 \) for all \( i = 1, 2, \ldots, n \). Thus \( x \in K^o \), and since \( x \) was arbitrary, \( ((K^o)^*)^* \subset K^o \). Conclude that \( ((K^o)^*)^* = K^o \).

12. Prove Eq.8.6:
\[
AK = \sum_{i=1}^{n} \bar{V}_i; \quad AK^o = \sum_{i=1}^{n} V_i,
\]
where \( A \in \mathbb{R}^{m \times n} \), and \( K, K^o \) are the orthants of Definition 6.

**Solution:** Recall that \( v_i \in \mathbb{R}^m \) is the \( i \)th column of \( A \), defining the rays \( \bar{V}_i = \{cv_i : c > 0\} \) and \( V_i = \{cv_i : c \geq 0\} \) for \( i = 1, \ldots, n \). Then \( x \in AK \) iff there exists \( k = (k_1, \ldots, k_n) \in K \) such that \( x = A k \). But then,
\[
x = A k = \sum_{i=1}^{n} k_i v_i \in \sum_{i=1}^{n} \bar{V}_i,
\]
since \( k_i v_i \in \bar{V}_i \) because \( k_i \geq 0 \) for all \( i \).
Likewise, \( x \in AK^o \) iff there exists \( k^o = (k_1^o, \ldots, k_n^o) \in K^o \) such that \( x = A k^o \). But then
\[
x = A k^o = \sum_{i=1}^{n} k_i^o v_i \in \sum_{i=1}^{n} V_i,
\]
since \( k_i^o v_i \in V_i \) because \( k_i^o > 0 \) for all \( i \).

13. Prove Corollary 8.18 on p.209: The set \( S \) of strictly profitable portfolios is a strict dual cone: \( S = (AK^o)^* \)

**Solution:** Modify the proof of Corollary 8.17 as follows:

**Proof:** Since \( (K \setminus 0)^* = K^o \) by Theorem 8.16(c),
\[
s \in S \iff s^T A \in K \setminus 0
\]
\[
\iff (\forall k \in (K \setminus 0)^*)(s^T A)k > 0
\]
\[
\iff (\forall k \in K^o)(s^T A)k > 0
\]
\[
\iff (\forall k \in K^o)s^T (Ak) > 0
\]
\[
\iff (\forall v \in AK^o)s^T v > 0
\]
\[
\iff s \in (AK^o)^*.
\]
since $AK^\circ = \{Ak : k \in K^\circ\}$, so that the next to last condition is just the definition of membership in the strict dual cone $(AK^\circ)^*$.  

14. Suppose $S \subset \mathbb{R}^n$ is any set. Prove the following:

(a) $S^\perp$ is a subspace.
(b) $S^* \subset S'$ and thus $S^* \cap S' = S^*$.
(c) $S^\perp \subset S'$ and thus $S^\perp \cap S' = S^\perp$.
(d) $S^\perp \cap S^* = \emptyset$.
(e) $S^\perp$, $S'$, and $S^*$ are all convex cones.
(f) If $0 \in S$, then $S^* = \emptyset$. Thus if $S$ is a subspace, then $S^* = \emptyset$.

**Solution:** (a) Check the definition:
- $0 \in S^\perp$ since $0^T s = 0$ for any $s \in S$.
- Given $x, y \in S^\perp$, take any $s \in S$ and compute
  $$s^T(x + y) = s^T x + s^T y = 0 + 0 = 0.$$  
  Conclude that $x + y \in S^\perp$.
- Given $x \in S^\perp$ and $c \in \mathbb{R}$, take any $s \in S$ and compute
  $$s^T(cx) = cs^T x = c0 = 0.$$  
  Conclude that $cx \in S^\perp$.

(b) $v^T s > 0 \implies v^T s \geq 0$, so every $v \in S^*$ also belongs to $S'$.
(c) $v^T s = 0 \implies v^T s \geq 0$, so every $v \in S^\perp$ also belongs to $S'$.
(d) It is impossible to have both $v^T s = 0$ and $v^T s > 0$, so there are no vectors $v$ in both $S^*$ and $S^\perp$.

(e) Check the two needed properties for $S^\perp$, $S'$, and $S^*$:

**Cones:** Let $x$ be a vector in $\mathbb{R}^n$ and let $\lambda$ be a positive real number.
- $(\forall s \in S)s^T x = 0 \implies (\forall s \in S)s^T (\lambda x) = \lambda 0 = 0$;
- $(\forall s \in S)s^T x \geq 0 \implies (\forall s \in S)s^T (\lambda x) = \lambda s^T x \geq 0$;
- $(\forall s \in S)s^T x > 0 \implies (\forall s \in S)s^T (\lambda x) = \lambda s^T x > 0$.

**Convex:** Let $x, y$ be vectors in $\mathbb{R}^n$ and let $\lambda \in [0, 1]$ be a real number. Let $s \in S$ be arbitrary.
- $s^T[\lambda x + (1 - \lambda)y] = \lambda s^T x + (1 - \lambda)s^T y = 0$ if $x, y \in S^\perp$;
- $s^T[\lambda x + (1 - \lambda)y] = \lambda s^T x + (1 - \lambda)s^T y \geq 0$ if $x, y \in S'$;
- $s^T[\lambda x + (1 - \lambda)y] = \lambda s^T x + (1 - \lambda)s^T y > 0$ if $x, y \in S^*$.
(f) If $0 \in S$, then every $x \in \mathbb{R}^n$ gives $x^T 0 = 0$, so there is no $x \in \mathbb{R}^n$ such that $x^T 0 > 0$, so $S^* = \emptyset$. \hfill \Box

15. Suppose that $n > 2$ and market model $A, q$ has

$$A = \begin{pmatrix} R & \cdots & R \\ a_1 & \cdots & a_n \end{pmatrix},$$

where $R > 1$ is the riskless return and $a = (a_1, \ldots, a_n)$ is a nonconstant payoff vector for the sole risky asset.

(a) Find necessary and sufficient conditions on $q$ such that $A, q$ is arbitrage-free. (Hint: use the Fundamental Theorem.)

(b) Exhibit a derivative payoff $d$ for which no exact hedge exists. (This shows that $A$ is not a complete market.)

(c) Exhibit a derivative $d$ for which an exact hedge does exist.

Solution: (a) Following the hint, observe that $A, q$ is arbitrage-free if and only if there exists a positive vector $k = [k_1, \ldots, k_n]^T$ such that $q = Ak$. But $q = [1, S_0]^T$, so $Rk$ is a p.d.f., so if $q = Ak$, then

$$S_0 = q(2) = Ak(2) = \sum_{i=1}^n a_i k_i = \frac{1}{R} \sum_{i=1}^n a_i (R k_i),$$

which is possible if and only if $\min\{a_i\} < RS_0 < \max\{a_i\}$. \hfill \Box

Remark. Thus $RS_0$ must be inside the range of the payoffs $\{a_i\}$.

(b) It may be assumed that $a_1 < a_2 \leq a_3$. (Otherwise, simply renumber the states.) Submatrix

$$A_2 = \begin{pmatrix} R & R \\ a_1 & a_2 \end{pmatrix}$$

is invertible, so the numbers $h_0, h_1$ are uniquely determined by

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = A_2^{-1} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where $d = [d_1, d_2, d_3, \ldots]^T$ is the payoff vector for the derivative to be hedged. Now choose $d_1 = d_2 = 1$, determine $h_0, h_1$, and choose $d_3$ different from $h_0 R + h_1 a_3$. The derivative with this payoff cannot be exactly hedged in this market.

(c) As in part (b), choose $d_1 = d_2 = 1$, determine $h_0, h_1$, but choose

$$d_i = h_0 R + h_1 a_i, \quad i = 3, \ldots, n.$$ 

The row vector $d = [1, d_3, \ldots, d_n]$ lies in the row space of $A$ and thus is exactly hedged by $h = [h_0, h_1]$. \hfill \Box

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16. Suppose that a market model has five states, a riskless asset returning $R = 1.02$, and two risky assets $a, b$ with spot prices $a_0 = 20$ and $b_0 = 12$ and payoffs $a = (10, 15, 20, 25, 30)$ and $b = (17, 15, 12, 10, 7)$, respectively.

(a) Prove that the model is arbitrage-free.

(b) Find the no-arbitrage bid-ask interval for a European-style Call option on $a$ with strike price 20.

(c) Find the no-arbitrage bid-ask interval for a European-style Put option on $b$ with strike price 13.

**Solution:** The computations may be done in Octave with the `glpk` package. Begin by putting the data into a finite market model $Aq$:

\[
a = [10, 15, 20, 25, 30]; \quad b = [17, 15, 12, 10, 7]; \quad a_0 = 20; \quad b_0 = 12; \\
R = 1.02; \quad A = [R R R R R; a; b]; \quad q = [1; a_0; b_0]; \quad Aq
\]

(This is in the notation of Definition 4.)

(a) To show that $A, q$ is arbitrage-free, by Fundamental Theorem 8.4 it suffices to find $k > 0$ such that $q = Ak$. This $3 \times 5$ linear system may be placed into row echelon form, yielding

\[
\text{rref}([A R*q])
\]

```
% 1.00000 0.00000 0.00000 -1.00000 -1.00000 -0.48000
% 0.00000 1.00000 0.00000 1.00000 -0.00000 0.88000
% 0.00000 0.00000 1.00000 1.00000 2.00000 0.60000
```

(Use $R*q$ in the augmented matrix to get the risk neutral probabilities $p = Rk$, which sum to 1, instead of the discounted vector $k$ which will sum to $\frac{1}{R}$.) The complete set of three pivot rows shows that the system is consistent but underdetermined and thus has a two-parameter family of solutions. By Corollary 8.8, it suffices to check that the unique minimal norm solution is positive. Octave computes it with

\[
A\backslash q*R \quad % p = 0.024 0.432 0.040 0.448 0.056 \\
A\backslash q \quad % k = 0.023529 0.423529 0.039216 0.439216 0.054902
\]

Conclude that $A, q$ is arbitrage-free.

**Remark.** The row echelon form of $[A R*q]$ shows how to find the complete solution set of positive vectors $Rk \overset{\text{def}}{=} p = (p_1, p_2, p_3, p_4, p_5)$. Identify the pivot variables $p_1, p_2, p_3$, so $p_4$ and $p_5$ are free variables. The general solution may be expressed as

\[
p_1 = p_4 + p_5 - 0.48 \\
p_2 = -p_4 + 0.88 \\
p_3 = -p_4 - 2p_5 + 0.60
\]
According to the given information, 
\[ \sum_i p_i = 1 \] 
to find a positive solution \( p > 0 \) requires solving the simultaneous inequalities:

\[ p_4 > 0, \; p_5 > 0, \; p_4 + p_5 > 0.48, \; p_4 < 0.88, \; p_4 + 2p_5 < 0.60, \]

which reduce to the intervals \( 0 < p_5 < 0.12 \) and \( 0.48 - p_5 < p_4 < 0.60 - 2p_5 \). One of the infinitely many solutions is thus parametrized by the midpoints:

\[
\begin{align*}
p_4 &= \frac{(0.48-p_5)+(0.60-2p_5)}{2}; \\
p_5 &= \frac{p_4+p_5-0.48; \; p_2 = -p4+0.88; \; p_3 = -p4-2p5+0.60; }{2}; \\
p &= [p_1 \; p_2 \; p_3 \; p_4 \; p_5]; \; p = 0.03 \; 0.43 \; 0.03 \; 0.45 \; 0.06; \\
k &= \frac{p}{R}; \; k = 0.029412 \; 0.421569 \; 0.029412 \; 0.441176 \; 0.058824.
\end{align*}
\]

This gives another explicit positive solution \( k > 0 \), proving that \( A, q \) is arbitrage-free without using Corollary 8.8.

For parts (b) and (c), reuse the Octave code from Section 8.2.3, first putting the parameters, market matrix and spot prices into GLPK format:

```
sellctype="LLLLL"; \% Lower constraint type \( A'*x(j) >= bb(j), j=1:5 \) 
sellsense=1; \% Optimization direction for \( q'*x \): \"1\" \Rightarrow \"min\" 
buyctype="UUUUU"; \% Upper constraint type \( A'*x(j) =< bb(j), j=1:5 \) 
buysense=-1; \% Optimization direction for \( q'*x \): \"-1\" \Rightarrow \"max\" 
vartype="CCC"; \% \( x(j) \) is Continuous, \( j=1:3 \) 
param.msglev=1; \% \Rightarrow use a low verbosity level 
huge=1000; \% huge and huger 
param.itlim=huge; \% \Rightarrow huge maximum number of iterations 
lb=[-infty; -huge; -huge]; \% huge Lower bounds on \( x \) 
ub=[ infty; huge; huge]; \% huge Upper bounds on \( x \)
```

(b) Compute the Call payoff on asset \( a \), then find a superreplication and a subreplication, taking the derivative seller’s and buyer’s perspectives, respectively:

\[
\begin{align*}
K_a &= 20; \; ba = \max(a-K_a,0); \% C(T): payoff for "a" Call 
[hs,ask]=glpk(q,A',ba,lb,ub,sellctype,vartype,sellsense,param) \% hs = [102.9412; -2.00; -5.00]; Cost-minimizing hedge portfolio 
% ask = 2.9412; minimum cost to superreplicate the Call 
[hb,bid]=glpk(q,A',ba,lb,ub,buyctype,vartype,buysense,param) \% hb = [-117.6471; 3.00; 5.00]; Cost-maximizing hedge portfolio 
% bid = 2.3529; maximum cost to subreplicate the Call 
\end{align*}
\]

Since bid is strictly less than ask, by Corollary 8.9 there is no exact hedge for this derivative in this market.

(c) Compute the Put payoff on asset \( b \), then find a superreplication and a subreplication, taking the derivative seller’s and buyer’s perspectives, respectively:

\[
\begin{align*}
K_b &= 20; \; ba = \max(b-K_b,0); \% C(T): payoff for "b" Call 
[hs,ask]=glpk(q,A',ba,lb,ub,sellctype,vartype,sellsense,param) \% hs = [102.9412; -2.00; -5.00]; Cost-minimizing hedge portfolio 
% ask = 2.9412; minimum cost to superreplicate the Call 
[hb,bid]=glpk(q,A',ba,lb,ub,buyctype,vartype,buysense,param) \% hb = [-117.6471; 3.00; 5.00]; Cost-maximizing hedge portfolio 
% bid = 2.3529; maximum cost to subreplicate the Call 
\end{align*}
\]
Kb=13; bb=max(Kb-b,0); % P(T): payoff for "b" Put
[hs,ask]=glpk(q,A',bb,lb,ub,sellctype,vartype,sellsense,param)
% hs = [ 61.7647; -1.20; -3.00]; Cost-minimizing hedge portfolio
% ask = 1.7647; minimum cost to superreplicate the Call
[hb,bid]=glpk(q,A',bb,lb,ub,buyctype,vartype,buysense,param)
% hb = [ -26.47059; 0.80; 1.00]; Cost-maximizing hedge portfolio
% bid = 1.5294; maximum cost to subreplicate the Call

Since bid is strictly less than ask, by Corollary 8.9 there is no exact hedge for this derivative in this market. □