

Math 456: Introduction to Financial Mathematics

Homework Set 3

Note: You may assume that the probability mass function is never zero in any finite probability space Ω , namely that $\Pr(\{\omega\}) > 0$ for every point $\omega \in \Omega$.

1. Suppose that $S(t, \omega)$, $0 \leq t \leq T$ is the price of a risky asset S , and that the riskless return over time $[0, T]$ is R . Model the future at time $t = T$ using $\Omega = \{\uparrow, \downarrow\}$ and assume that $S(T, \downarrow) < S(T, \uparrow)$.

(a) Use the no-arbitrage Axiom 1 to conclude that

$$S(T, \downarrow) < RS(0) < S(T, \uparrow).$$

(b) Use the Fair Price Theorem 1.4 to prove the same inequalities.

2. In Exercise 1 above, model the future at time $t = T$ using the N -step binomial model $\Omega = \{\omega_0, \omega_1, \dots, \omega_N\}$ and assume that

$$S(T, \omega_k) = S(0)u^k d^{N-k},$$

where $S(0) > 0$ is the spot price and $0 < d < u$ are the up factor and down factor, respectively, over one time step T/N .

(a) Use the no-arbitrage Axiom 1 to conclude that

$$d < R^{1/N} < u.$$

(b) Use the Fair Price Theorem 1.4 to prove the same inequalities.

3. Suppose that a portfolio X contains risky stock S and riskless bond B in amounts h_0, h_1 :

$$X(t, \omega) = h_0 B(t, \omega) + h_1 S(t, \omega).$$

Model the future at time $t = T$ using $\Omega = \{\uparrow, \downarrow\}$, assuming only that $S(T, \uparrow) \neq S(T, \downarrow)$ and that $B(T, \uparrow) = B(T, \downarrow) = R$. Compute h_0 and h_1 in terms of all the other quantities. (Hint: use Macsyma to derive Eq.3.1.)

4. In Exercise 3 above, suppose that X is a European-style Call option for S with expiry T and strike price K . Use the payoff formula $X(T) = [S(T) - K]^+$ in the equation for h_1 to prove that

$$0 \leq h_1 \leq 1.$$

Conclude that, in this model of the future, a European-style Call option for S is equivalent to a portfolio containing part of a share of S plus or minus some cash.

5. In Exercise 3 above, suppose that X is a European-style Put option for S with expiry T and strike price K . Use the payoff formula $X(T) = [K - S(T)]^+$ in the equation for h_1 to prove that

$$-1 \leq h_1 \leq 0.$$

Conclude that, in this model of the future, a European-style Put option for S is equivalent to a portfolio containing part of a share of S sold short plus or minus some cash.

6. Suppose that $C(0)$ and $P(0)$ are the premiums for European-style Call and Put options, respectively, on an asset S with the following parameters: expiry at $T = 1$ year, spot price $S(0) = 90$, strike price $K = 95$. Assume that the riskless annual percentage rate is $r = 0.02$, and the volatility for S is $\sigma = 0.15$, and that these will remain constant from now until expiry.
- Use a LibreOffice Calc spreadsheet to implement the Cox-Ross-Rubinstein (CRR) model to compute $C(0)$ and $P(0)$ with $N = 10$ time steps, using the backward pricing formula in Eq.3.18. (Hint: compare output with `CRReurAD()` to check for bugs.)
 - Use the Octave function `CRReurAD()` with $N = 10$, $N = 100$, and $N = 1000$ time steps to compute $C(0)$ and $P(0)$.
 - Repeat part (b) with the Octave function `CRReur()` on p.88, again using $N = 10$, $N = 100$, and $N = 1000$ time steps to compute $C(0)$ and $P(0)$. Profile the time required to compute them, and compare the time and the output with that of `CRReurAD()`.
 - Compare the prices from parts (b) and (c). Is it justified to use $N = 1000$? Is $N = 10$ sufficiently accurate?
7. Compare the prices from parts (a) and (b) of previous Exercise 6 with the Black-Scholes prices computed using Eqs.2.25 and 2.26. Plot the logarithm of the differences against $\log N$ to estimate the rate of convergence. (Hint: Use the programs in Chapter 2, Section 2.4.)
8. Derive 3.32 on p.79:

$$q = \frac{1}{2} + \frac{r + \frac{\sigma^2}{2}}{2\sigma} \sqrt{\frac{T}{N}} + O\left(\frac{T}{N}\right).$$

9. Use the CRR approximation with $N = 4$ to compute the European-style Call option premiums at several hundred equally spaced spot prices $75 \leq S_0 \leq 115$, with expiry $T = 1$, strike $K = 95$, $r = 0.02$, and $\sigma = 0.15$.
- Plot the values against S_0 .
 - At what values of S_0 in that range does the graph appear to be nonsmooth?
 - Compute the points of nondifferentiability for S_0 in $[75, 115]$.

10. Compute the CRR option premiums and Greeks for European-style Call and Put options on a risky asset with the following parameters: spot price \$90, strike price \$95, expiry in 1 year, annual riskless rate 2%, and volatility 15%. Use $N = 100$ steps. Justify the method used.