Math 456: Introduction to Financial Mathematics

Homework Set 1

1. Suppose that a risky asset S has spot price $S(0) = 100$ and that the riskless return to $T = 0.5$ year is $R = 1.0225$. Assuming there are no arbitrages, compute the following:

   (a) the current zero-coupon bond discount $Z(0, T)$,

   (b) the Forward price for one unit of S at expiry T,

   (c) the riskless annual interest rate (assuming continuous compounding),

   Solution:

   (a) From the formula on p.3, $Z(0, T) = \frac{1}{R} = 0.9780$.

   (b) By Eq.1.15, the fair price is $K = RS(0) = 102.25$.

   (c) By Eq.1.6, $r = (\log R)/T = 0.0445$, or 4.45%. $\square$

2. With $S(0)$, R, and T as in Exercise 1, suppose that $S(T)$ is modeled by

$S(T)$	93	95	99	101	103	105	107	109
$\Pr(S(T))$	0.03	0.06	0.09	0.31	0.33	0.09	0.06	0.03

   (a) What is the expected future price $S(T)$ in this finite probability space model?

   (b) Use this finite probability space model to estimate premiums $C(0)$ and $P(0)$ for European-style Call and Put options, respectively, with strike price $K = 102$ and expiry T.

   (c) Does Call-Put Parity hold in this model? What might cause it to be inaccurate?

   Solution:

   First enter the data with these Octave/Matlab commands:

```
R=1.0225; S0=100; ST=[93 95 99 101 103 105 107 109];
K=102; Pr=[0.03 0.06 0.09 0.31 0.33 0.09 0.06 0.03];
```

   (a) By Eq.1.11, in its discrete form on p.10, the expected price at expiry is

$$E(S(T)) = \sum_{\omega \in \Omega} \Pr(\{\omega\}) S(T, \omega)$$

   which may be computed with Octave/Matlab commands

```
EST=Pr*ST' % EST =  101.84
```

(b) By Eq.1.8, the Call payoff at expiry is $C(T) = [S(T) - K]^+$. By Fair Price Theorem 1.4, the no-arbitrage Call premium is the present value of the expected value of this payoff:

$$C(0) = \mathrm{E}(C(T))/R = \frac{1}{R} \sum [S(T) - K]^+ \, \mathrm{Pr}(S(T)),$$

which may be computed with Octave/Matlab commands

```
Cpayoff=max(0,ST-K); C0=(Cpayoff*Pr')/R  % C0 =  1.0856
```

Likewise, by Eq.1.9 the Put payoff is $P(T) = [K - S(T)]^+$ and the no-arbitrage Put premium is therefore:

$$P(0) = \mathrm{E}(P(T))/R = \frac{1}{R} \sum [K - S(T)]^+ \, \mathrm{Pr}(S(T)),$$

which may be computed with the commands

```
Ppayoff=max(0,K-ST); P0=(Ppayoff*Pr')/R  % P0 = 1.2421
```

(c) Observe that Call-Put parity does not quite hold:

```
C0-P0    %  ans = -0.1565
S0-K/R   %  ans = 0.2445
```

The difference is approximately 0.40 instead of exacly 0 as in Eq.1.16. This is roughly the difference 0.41 between $E(S(T)) = 101.84$ computed in part (a), which is the model's estimate for the Forward price of S, and $RS(0) = 102.25$, which is another Forward estimate using riskless return. It seems that the model is overly pessimistic about the asset's price at future time T. □

3. Use the no arbitrage Axiom 1 to prove that Eq.1.7 holds.

   Solution:

   If the asset A costs more, then short-sell A for $A(0)$, buy the sequence of zero-coupon bonds, pocket the surplus, and use the proceeds from the bonds to pay the cash flow to the A buyer over time, settling all liabilities.

   Otherwise if the asset A costs less, short-sell the bonds, use the money to buy A for $A(0)$, pocket the surplus, and then pay the bond buyers back over time to settle all obligations. In either case there is an arbitrage. □

4. Prove Corollary 1.3. (Hint: review the proof of Theorem 1.2.)

   Solution: Recall the statement of Corollary 1.3: If at some time $T > 0$, $A(T, \omega) > B(T, \omega)$ in all states ω, then $A(t) > B(t)$ for all times $0 \leq t < T$.

Proof: Suppose not. Then there is a time t with $0 \le t < T$ such that $A(t) \le B(t)$, so at that time it is possible to assemble the portfolio $-A+B$ by selling B short and buying A. Observe that $-A(t) + B(t) \le 0$, so this portfolio costs nothing and may even generate surplus cash.

At time T, sell A for $A(T, \omega)$ and buy B for $B(T, \omega)$ to cover the short. By hypothesis this pays off $A(T, \omega) - B(T, \omega) > 0$ in all states ω. The assembled portfolio would thus be an arbitrage opportunity, contradicting Axiom 1.

Conclude that $A(t) > B(t)$ for all times $0 \le t \le T$. □

5. Suppose, in contradiction with Eq.1.16, that $C(0) - P(0) < S(0) - K/R$. Construct an arbitrage.

   Solution: At $t = 0$, starting with no money, do the following:

   - Short-sell P for $P(0)$ cash.
   - Short-sell S for $S(0)$ cash.
   - Buy C for $C(0)$ cash.
   - Deposit K/R cash into the bank at riskless return R.

   That leaves $S(0) - K/R - C(0) + P(0) > 0$ cash. Then at $t = T$, clear all debts as follows:

   - If $S(T) > K$ then exercise C for a profit of $S(T) - K$. Otherwise C expires worthless.
   - If $K > S(T)$, so cash-settle the short-sold P for $K - S(T) > 0$. Otherwise P expires worthless.
   - Withdraw $RK/R = K$ cash from the bank, including interest.
   - Cover the short sale of S for $S(T)$ cash.

   That leaves $C(T) - P(T) + K - S(T) = [S(T) - K]^+ - [K - S(T)]^+ + K - S(T) = 0$ and no debts in all cases. The positive amount obtained at $t = 0$ is therefore an arbitrage profit forbidden by Axiom 1. □

6. Prove Eq.1.20, the Call-Put parity Formula for foreign exchange options:

$$C(0) - P(0) = \frac{X(0)}{R_f} - \frac{K}{R_d},$$

   using the no arbitrage axiom.

   Solution: To prove equality, show that inequality in either direction results in an arbitrage opportunity.

   Case 1: $C(0) - P(0) > X(0)/R_f - K/R_d$.

   At $t = 0$, starting with no money, perform the following trades:

   - Short-sell C for $C(0)$ DOM cash.

- Buy P for $P(0)$ DOM cash.
- Borrow K/R_d DOM cash at R_d from the domestic bank.
- Convert $X(0)/R_f$ DOM to $1/R_f$ FRN at spot exchange rate $X(0)$.
- Deposit $1/R_f$ FRN cash into the foreign bank at R_f.

That leaves $C(0)-P(0)-X(0)/R_f+K/R_d > 0$ DOM in cash, a net profit to keep.

At $t = T$, clear all debts as follows:

- Withdraw 1 FRN cash from the foreign bank, including interest.
- If $K \geq X(T)$, then
  - Short-sold C expires worthless, as $C(T) = [X(T) - K]^+ = 0$ imposes no liabilty.
  - Exercise P to convert 1 FRN into K DOM.
- Else $K < X(T)$, so
  - Convert 1 FRN to $X(T)$ DOM at the market rate $X(T)$.
  - Cash settle C for $C(T) = [X(T) - K]^+ = X(T) - K$.
  - Do not exercise P, as $P(T) = [K - X(t)]^+ = 0$ is worthless.
- Repay the domestic bank loan with interest for K DOM cash.

That leaves $-[X(T) - K]^+ + [K - X(T)]^+ - K + X(T) = 0$ and no unfunded liabilities. The positive amount obtained at $t = 0$ is therefore an arbitrage profit prohibited by Axiom 1.

Case 2: $C(0) - P(0) < X(0)/R_f - K/R_d$.

At $t = 0$, starting with no money, perform the following trades:

- Short-sell P for $P(0)$ cash.
- Buy C for $C(0)$ cash.
- Borrow $1/R_f$ FRN from the foreign bank at R_f.
- Convert $1/R_f$ FRN to $X(0)/R_f$ DOM at spot exchange rate $X(0)$.
- Deposit K/R_d DOM cash into the domestic bank.

That leaves $X(0)/R_f - K/R_d - C(0) + P(0) > 0$ DOM cash, a net profit to keep.

At $t = T$, clear all debts as follows:

- Withdraw K DOM cash from the domestic bank, including interest.
- If $K \geq X(T)$, then
  - Do not exercise the worthless C, as $C(T) = [X(T) - K]^+ = 0$.
  - Cash settle P for its value $P(T) = [K - X(T)]^+ = K - X(T)$.
  - Convert $X(T)$ DOM to 1 FRN cash at the market rate $X(T)$.
- Else $K < X(T)$, so
  - Short-sold P expires worthless, as $P(T) = [K - X(T)]^+ = 0$.

- Exercise C to convert K DOM to 1 FRN cash at rate K.

- Repay the foreign bank loan with interest for 1 FRN cash.

That leaves $[K - X(T)]^{+} - [X(T) - K]^{+} + K - X(T) = 0$ and no unfunded liabilities. The positive amount obtained at $t = 0$ is therefore an arbitrage profit prohibited by Axiom 1.

Conclude that Eq.1.20 holds. □

7. (a) Prove that the plus-part function satisfies Eq.1.17:

$$[X]^{+} - [-X]^{+} = X,$$

for any number X.

(b) Apply the identity in part (a) to the payoff values of European-style Call and Put options for S at strike price K and expiry T to show Eq.1.18:

$$C(T) - P(T) = S(T) - K.$$

Solution:

(a) Let X be any number. Then either $X < 0$, $X = 0$, or $X > 0$. Check all three possibilities:

$$[X]^{+} - [-X]^{+} = \begin{cases} 0 - (-X) = X, & X < 0, \\ 0 + 0 = X, & X = 0, \\ X - 0 = X, & X > 0, \end{cases} \quad = \quad X,$$

in all cases.

(b) Let $X = S(T) - K$. Then $C(T) = [S(T) - K]^{+} = [X]^{+}$ while $P(T) = [K - S(T)]^{+} = [-X]^{+}$. Apply part (a) to conclude that

$$C(T) - P(T) = [X]^{+} - [-X]^{+} = X = S(T) - K,$$

as claimed. □

8. Plot the payoff and profit graphs for the following colorfully named option portfolios as a function of the price $S(T)$ at expiry time T:

(a) *Long straddle:* buy one Call and one Put on S with the same expiry T and at-the-money strike price $K \approx S(0)$. For what values of $S(T)$ will this be profitable?

(b) *Long strangle:* buy one Call at K_c and one Put at K_p with the same expiry T but with out-of-the-money strike prices $K_p < S(0) < K_c$. How does its profitability compare with that of a long straddle?

Solution:

(a) See the payoff graph for a Long straddle in Figure 1.

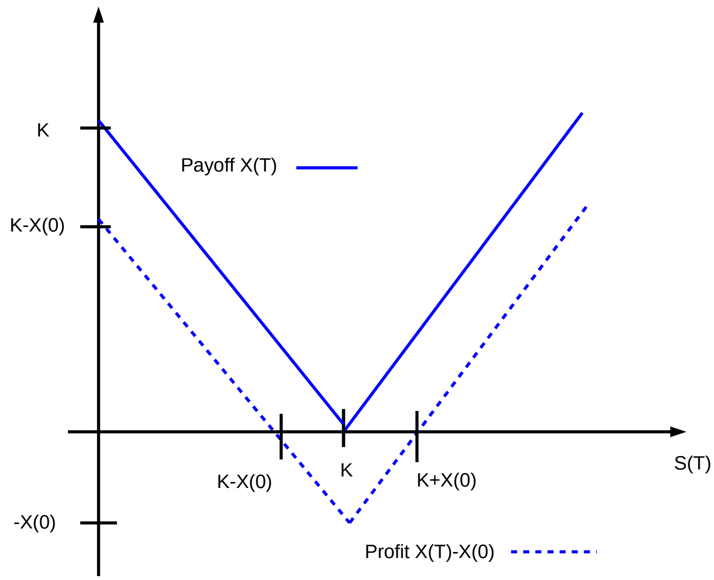

Figure 1: Payoff and profit for a Long straddle portfolio $X = C + P$.

The straddle will be profitable if $S(T)$ is sufficiently far from $S(0)$ in either direction, namely more than the sum of the two premiums.

(b) See the payoff graph for a Long strangle in Figure 2.

Since the two strangle options are out-of-the-money, their premiums are lower than the at-the-money options in a straddle, making the strangle portfolio cheaper. However, for the strangle to be profitable, the stock price may have to move farther. □

9. A *butterfly spread* is a portfolio of European-style Call options purchased at time $t = 0$ with the same expiry $t = T$ but at three strike prices $L < M < H$, where $M = \frac{1}{2}(L + H)$:

   - buy one Call C_L at strike price L for $C_L(0)$,
   - buy one Call C_H at strike price H for $C_H(0)$,
   - sell two Calls C_M short at strike price M for $2C_M(0)$.

   (a) Plot the payoff graph for the butterfly spread at expiry when its price is $C_L(T) + C_H(T) - 2C_M(T)$. Mark the three strike prices on the $S(T)$ axis.

   (b) Conclude from the graph for (a) that $C_M(0) < \frac{1}{2}[C_L(0) + C_H(0)]$. (Hint: use the no arbitrage Axiom 1.)

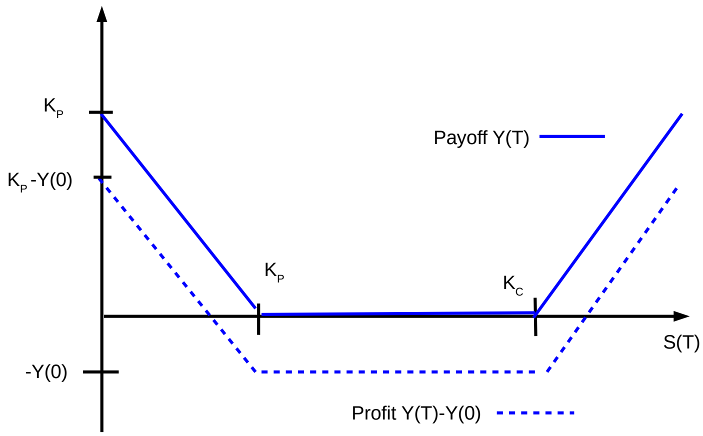

Figure 2: Payoff and profit for a Long strangle portfolio $Y = C_{K_C} + P_{K_P}$.

Solution: (a) See the payoff graph for a butterfly spread in Figure 3.

(b) The net cost of a butterfly spread at $t = 0$ is $2C_M(0) - C_L(0) - C_H(0)$. If $C_M(0) \geq \frac{1}{2}[C_L(0) + C_H(0)]$ then this net cost is greater than or equal to zero. But its value at $t = T$ is nonnegative in all states and positive in some states, namely if $L < S(T) < H$, so this is an arbitrage. Conclude by the no arbitrage Axiom 1 that $C_M(0) < \frac{1}{2}[C_L(0) + C_H(0)]$. $\square$

10. An *iron condor* is a portfolio $C_1 - C_2 - P_3 + P_4$ of four European-style options. To construct it, simultaneously buy one Call at K_1, sell one Call at K_2, sell one Put at K_3, and buy one Put at K_4, all with the same expiry T but with $K_1 < K_2 < K_3 < K_4$.

(a) Plot or describe the payoff graph for an iron condor portfolio at expiry.

(b) Assuming no arbitrage, prove that the portfolio must have a positive net premium.

(c) Assuming no arbitrage, find inequalities bounding the maximum profit and the maximum loss of an iron condor portfolio at expiry.

Solution:

(a) Let $icp(S(T))$ be the payoff at expiry T of the iron condor portfolio,

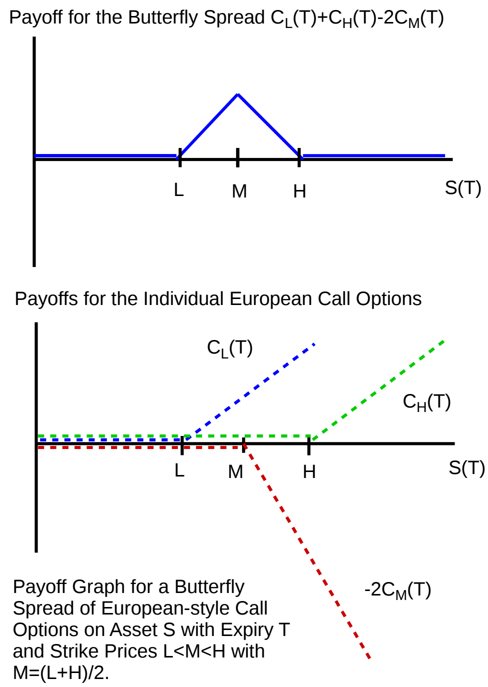

Figure 3: Payoffs for the butterfly spread $C = C_L + C_H - 2C_M$ and its constituent parts.

as a function of the underlying asset's price $S(T)$. By Eqs.1.8 and 1.9,

$$\begin{aligned}
icp(s) &= C_1(T) - C_2(T) - P_3(T) + P_4(T) \\
&= [s - K_1]^+ - [s - K_2]^+ - [K_3 - s]^+ + [K_4 - s]^+ \\
&= \begin{cases}
(K_4 - K_3), & s \le K_1, \\
(s - K_1) + (K_4 - K_3), & K_1 < s < K_2, \\
(K_2 - K_1) + (K_4 - K_3), & K_2 \le s \le K_3, \\
(K_2 - K_1) + (K_4 - s), & K_3 < s < K_4, \\
(K_2 - K_1), & s \ge K_4.
\end{cases}
\end{aligned}$$

(b) From part (a), the payoff is strictly positive. By Corollary 1.3, the net premium must therefore be positive.

(c) Let $y = C_1(0) + P_4(0) - C_2(0) - P_3(0)$ be the net premium for the iron condor portfolio. By part (b) it must be positive, and it must be subtracted from the payoff to give the profit at expiry. Hence the profit must be less than $(K_2 - K_1) + (K_4 - K_3)$.

The minimum net value at expiry, which is the worst possible loss, will be $\min\{(K_2{-}K_1), (K_4{-}K_3)\}{-}y$. But y is bounded above by the no arbitrage assumption since if $y > (K_2{-}K_1){+}(K_4{-}K_3)$, then the profit graph will lie entirely below the x-axis. Such an asset is guaranteed to lose, so selling it short would be an arbitrage opportunity. Hence $y \leq (K_2{-}K_1){+}(K_4{-}K_3)$, so the loss is bounded below by

$$\min\{(K_2{-}K_1), (K_4{-}K_3)\} - y \geq -\max\{(K_2{-}K_1), (K_4{-}K_3)\}.$$

Equality holds for $S(T) \leq K_1$ if $K_4{-}K_3$ is bigger, or for $S(T) \geq K_4$ if $K_2{-}K_1$ is bigger. $\square$