Ma 5052: Measure Theory and Functional Analysis II
Final and Qualifying Examination

Prof. Wickerhauser

$10+5$ *problems on* $1+2+1$ *pages*

Friday, May 6th, 2016

You have two hours to complete Part I of this test, problems 1–10 on pages 2 and 3, which is the final examination for Ma 5052.

You have one additional hour to complete Part II of this test, problems 11–15 on page 4, if you choose; the two parts together constitute the qualifying examination in Real Analysis.

No electronic devices may be used, nor any reference materials.

Please write your complete solutions in the bluebooks provided.

NOTE: you may quote without proof any theorem proved in class or in the homework sets, if you provide a precise statement of the result you are using.

PART I: Final examination for Math 5052.

1. Suppose $\{a_n : n = 1, 2, \ldots\}$ is a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n b_n < \infty$ whenever $\sum_{n=1}^{\infty} b_n^2 < \infty$. Prove that $\sum_{n=1}^{\infty} a_n^2 < \infty$.

2. Suppose that X is a metric space.

   (a) Prove that X is *first countable*.

   (b) Suppose in addition that X is *second countable*. Prove that every open covering of X has a countable subcovering.

   (c) Find, with proof, a metric space X that is not second countable.

3. Let $A \stackrel{\text{def}}{=} \text{span}\,\{f_n(x) \stackrel{\text{def}}{=} e^{inx} : n \in \mathbf{Z}\}$ be a set of complex-valued functions on $\mathbf{R}$.

   (a) Prove that A is dense in $C([0, 6])$.

   (b) Find a function in $C([0, 7])$ that is not in the closure of A in the supremum norm.

4. Let ℓ^2 be the Hilbert space of sequences $x = (x_1, x_2, \ldots)$ with inner product and derived norm

   $$\langle x, y\rangle \stackrel{\text{def}}{=} \sum_n x_n \bar{y}_n; \qquad \|x\| \stackrel{\text{def}}{=} \sqrt{\sum_n |x_n|^2},$$

   respectively. Suppose $T : \ell^2 \to \ell^2$ is the shift operator:

   $$T(x_1, x_2, \ldots) \stackrel{\text{def}}{=} (0, x_1, x_2, \ldots).$$

   (a) Find the spectral radius of T.

   (b) Compute the spectrum of T.

   (c) Determine, with proof, whether T is a compact operator.

5. Suppose A is a continuous linear operator on a Hilbert space H.

   (a) Prove that there exists a unique continuous linear operator A^* satisfying

   $$(\forall x \in X)(\forall y \in Y)\ \langle Ax, y\rangle = \langle x, A^* y\rangle\,.$$

   (b) Show that $\ker A \stackrel{\text{def}}{=} \{x \in H : Ax = 0\}$ and $\text{img}\,A^* \stackrel{\text{def}}{=} \{A^* x : x \in H\}$ are closed subspaces satisfying $\ker A = (\text{img}\,A^*)^{\perp}$.

6. (a) Give a precise statement of Plancherel's theorem.

   (b) If A is a Lebesgue measurable subset of $[0, 2\pi)$, prove that $\lim_{n\to\infty} \int_A e^{inx}\, dx = 0$.

7. Suppose $S : C([0,1]) \to C([0,1])$ is defined by $Sf(x) \stackrel{\text{def}}{=} \int_0^x f(y)\, dy$.

   (a) Prove that S is a compact linear operator.

   (b) State the spectral theorem for compact operators.

   (c) Compute the spectrum of S.

8. Prove the following generalization of Hölder's inequality: if $p_1, \ldots, p_m \in [1, \infty)$ and $q \geq 1$ satisfies

$$\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{p_i},$$

   then for all measurable functions $f_1, \ldots, f_m$,

$$\| \prod_i f_i \|_q \leq \prod_i \|f_i\|_{p_i}.$$

9. For $x \in \mathbf{R}$ and $n \in \mathbf{N}$, denote by $\delta_x^{(n)}$ the mapping

$$\delta_x^{(n)}(\phi) \stackrel{\text{def}}{=} \phi^{(n)}(x),$$

   where $\phi \in C_K^\infty(\mathbf{R})$ and $\phi^{(n)}$ denotes the nth derivative.

   (a) Prove that $\delta_x^{(n)}$ is a compactly supported distribution.

   (b) Put $F \stackrel{\text{def}}{=} \sum_{n=0}^\infty \delta_n^{(n)}$. Prove that F is a distribution.

   (c) Determine, with proof, whether F as defined in (b) is a tempered distribution.

10. Suppose that X is a random variable.

    (a) Prove *Chebyshev's inequality*:

$$(\forall a > 0)\ \Pr(|X| \geq a) \leq \frac{E|X|}{a}.$$

    (b) Prove that for all $a > 0$, $\Pr(|X - EX| \geq a) \leq \frac{\operatorname{Var} X}{a^2}$.

PART II: Additional problems for the Real Analysis Qualifying Examination.

11. Let $\{f_n\}$ be a sequence of measurable functions. Prove that the set of points x such that $\{f_n(x)\}$ *fails to converge* as $n \to \infty$ is measurable.

12. (a) Give a precise statement of Lebesgue's dominated convergence theorem for complex-valued functions.

    (b) Let μ be a finite measure on the Borel subsets of $\mathbf{R}$. Prove that the *Fourier transform* of the measure μ, namely the function

    $$f(x) \stackrel{\text{def}}{=} \int_{\mathbf{R}} e^{ixt}\, d\mu(t),$$

    is continuous and bounded.

13. (a) Give a precise statement of Fubini's theorem for complex-valued functions.

    (b) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable. Prove the equality

    $$\int_{\mathbf{R}} |f(x)|\, dx = \int_0^\infty m(\{x : |f(x)| \geq t\})\, dt,$$

    where m is Lebesgue measure.

14. Suppose that μ is a finite measure on the Borel subsets of $\mathbf{R}$.

    (a) Give a precise definition of *absolute continuity* of μ with respect to Lebesgue measure.

    (b) Prove that μ is absolutely continuous with respect to Lebesgue measure if and only if there exists a function $f \in L^1(\mathbf{R}, dx)$ such that

    $$\mu(A) = \int_A f(x)\, dx$$

    for all Borel subsets $A \subset \mathbf{R}$.

15. Let X be a compact metric space and let $C(X)$ be the normed linear space of continuous functions $f : X \to \mathbf{R}$, using the norm

    $$\|f\| \stackrel{\text{def}}{=} \sup\{|f(x)| : x \in X\}.$$

    (a) Prove that $C(X)$ is a Banach space.

    (b) Prove that every continuous linear functional $I : C(X) \to \mathbf{R}$ may be written as

    $$I(f) = \int_X f(x)\, d\mu,$$

    where μ is a finite signed measure on the Borel subsets of X.