Math 5052
Measure Theory and Functional Analysis II
Homework Assignment 7

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Due Friday, February 5th, 2016

Please do Exercises 3, 6, 14, 16*, 17, 18, 21*, 23*, 24, 27*.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. Some of these exercises originate from that source.

1. Find a measure space $(X, \mathcal{A}, \mu)$, a subspace Y of $L^1(X, \mu)$, and a bounded linear functional f on Y with norm 1 such that f has two distinct extensions to $L^1(X, \mu)$ and each of the extensions has norm equal to 1.

2. Show that if $1 \le p < \infty$, then $L^p([0,1])$ is *separable*, namely that there is a countable dense subset.

3. Show that $L^\infty([0,1])$ is not separable, namely that any dense subset must be uncountable.

4. For $k \ge 1$ and functions $f : [0,1] \to \mathbf{R}$ that are k times differentiable, define

$$\|f\|_{C^k} \stackrel{\text{def}}{=} \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(k)}\|_\infty,$$

   where $f^{(k)}$ is the kth derivative of f. Let $C^k([0,1])$ be the collection of k times continuously differentiable functions f with $\|f\|_{C^k} < \infty$.

   Is $C^k([0,1])$ complete with respect to the norm $\|\cdot\|_{C^k}$?

5. Fix $\alpha \in (0,1)$. For a continuous function $f : [0,1] \to \mathbf{R}$, define

$$\|f\|_{C^\alpha} \stackrel{\text{def}}{=} \sup_{x \in [0,1]} |f(x)| + \sup_{x \ne y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

   Let $C^\alpha([0,1])$ be the set of continuous functions f with $\|f\|_{C^\alpha} < \infty$.

   Is $C^\alpha([0,1])$ complete with respect to the norm $\|\cdot\|_{C^\alpha}$?

6. For positive integers n, let

$$A_n \stackrel{\text{def}}{=} \left\{ f \in L^1([0,1]) : \int_0^1 |f(x)|^2 \, dx \le n \right\}.$$

Show that each A_n is a closed subset of $L^1([0,1])$ with empty interior.

7. Suppose L is a linear functional on a normed linear space X. Prove that L is a bounded linear functional if and only if the set $Z \stackrel{\text{def}}{=} \{x \in X : Lx = 0\}$ is closed.

8. Suppose X and Y are Banach spaces and $\mathcal{L}$ is the collection of bounded linear maps from X into Y, with the usual operator norm:

$$\|L\| \stackrel{\text{def}}{=} \sup_{\|x\|_X \le 1} \|Lx\|_Y.$$

Define $(L+M)x \stackrel{\text{def}}{=} Lx + Mx$ and $(cL)x = c(Lx)$ for $L, M \in \mathcal{L}$, $x \in X$, and scalar c.

Prove that $\mathcal{L}$ is a Banach space.

NOTE: see Remark 18.10 on textbook p.178.

9. Set A in a normed linear space is called *convex* if

$$\lambda x + (1-\lambda) y \in A$$

whenever $x, y \in A$ and $\lambda \in [0,1]$.

a. Prove that if A is convex, then the closure of A is convex.

b. Prove that the open unit ball in a normed linear space is convex. (The open unit ball is the set of x such that $\|x\| < 1$.)

10. The unit ball in a normed linear space V is called *strictly convex* if $\|\lambda f + (1-\lambda) g\| < 1$ whenever $\|f\| = \|g\| = 1$, $f \ne g \in V$, and $\lambda \in (0,1)$.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

a. Prove that, if $1 < p < \infty$, then the unit ball in $L^p(X,\mu)$ is strictly convex.

b. Prove that if X contains two or more points, then the unit balls in $L^1(X,\mu)$ and $L^\infty(X,\mu)$ are not strictly convex.

11. Let X be a metric space containing two or more points. Prove that the unit ball in $\mathcal{C}(X)$ is not strictly convex.

12. Let f_n be a sequence of continuous functions on $\mathbf{R}$ that converge at every point. Prove that for every compact subset $K \subset \mathbf{R}$ there exists a number M such that $\sup_n |f_n|$ is bounded by M on that interval.

13. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a vector space X such that $\|x\|_1 \le \|x\|_2$ for all $x \in X$, and suppose X is complete with respect to both norms. Prove that there exists a positive constant c such that

$$\|x\|_2 \le c\|x\|_1$$

for all $x \in X$.

14. Suppose X and Y are Banach spaces.

a. Let $X \times Y$ be the set of ordered pairs (x, y), $x \in X$, $y \in Y$, with componentwise addition and multiplication by scalars. Define

$$\|(x,y)\|_{X\times Y} \stackrel{\text{def}}{=} \|x\|_X + \|y\|_Y.$$

Prove that $X \times Y$ is a Banach space.

b. Let $L : X \to Y$ be a linear map such that if $x_n \to x$ in X and $Lx_n \to y$ in Y, then $y = Lx$. Such a map is called a *closed map*. Let G be the *graph* of L, defined by

$$G \stackrel{\text{def}}{=} \{(x,y) \in X \times Y : y = Lx\}.$$

Prove that G is a closed subset of $X \times Y$, hence is complete.

c. Prove that the function $(x, Lx) \mapsto x$ is continuous, injective, linear, and surjective from G onto X.

d. Prove the *closed graph theorem*: If L is a closed linear map from one Banach space to another (and hence by part b has a closed graph), then L is a continuous map.

15. Let X be the space of continuously differentiable functions on $[0,1]$ with the supremum norm and let $Y = C([0,1])$. Define $D : X \to Y$ by $Df = f'$. Show that D is a closed map but not a bounded one.

16. Let A be the set of real-valued continuous functions on $[0,1]$ such that

$$\int_0^{1/2} f(x)\,dx - \int_{1/2}^1 f(x)\,dx = 1.$$

Prove that A is a closed convex subset of $C([0,1])$, but there does not exist $f \in A$ such that $\|f\| = \inf_{g\in A}\|g\|$.

17. Let A_n be the subset of the real-valued continuous functions on $[0,1]$ given by

$$A_n \stackrel{\text{def}}{=} \{f : (\exists x \in [0,1])(\forall y \in [0,1])|f(x) - f(y)| \le n|x-y|\}.$$

a. Prove that A_n is nowhere dense in $C([0,1])$.

b. Prove that there exist functions $f \in C([0,1])$ which are *nowhere differentiable* on $[0,1]$, namely $f'(x)$ does not exist at any point $x \in [0,1]$.

18. Let X be a linear space and let $E \subset X$ be a convex set with $0 \in E$. Define a non-negative function $\rho : X \to \mathbf{R}$ by

$$\rho(x) \stackrel{\text{def}}{=} \inf\{t > 0 : t^{-1}x \in E\},$$

with the convention that $\rho(x) = \infty = \inf \emptyset$ if no $t > 0$ gives $t^{-1}x \in E$. This called the *Minkowski functional* defined by E.

a. Show that ρ is a *sublinear functional*, namely it satisfies $\rho(0) = 0$, $\rho(x+y) \leq \rho(x) + \rho(y)$, and $\rho(\lambda x) = \lambda \rho(x)$ for all $x, y \in X$ and all $\lambda > 0$.

b. Suppose in addition that X is a normed linear space and E is an *open* convex set containing 0. Prove that the Minkowski functional defined by E is finite at every $x \in X$ and that $x \in E$ if and only if $\rho(x) < 1$.

19. Let X be a linear space and let $\rho : X \to \mathbf{R}$ be a sublinear functional that is finite at every point. Prove that

$$|\rho(x) - \rho(y)| \leq \max(\rho(y-x), \rho(x-y))$$

for every $x, y \in X$.

20. Let X be a normed linear space, let $E \subset X$ be an open convex set containing 0, and let $\rho : X \to \mathbf{R}$ be the Minkowski functional defined by E. (See exercise 18 part a.) Prove that ρ is continuous on X.

21. Let X be a linear space and let $\rho : X \to \mathbf{R}$ be a sublinear functional. Suppose that M is a subspace of X and $f : M \to \mathbf{R}$ is a linear functional dominated by ρ, namely

$$f(x) \leq \rho(x), \qquad x \in M.$$

Prove that there exists a linear functional $F : X \to \mathbf{R}$ that satisfies $F(x) = f(x)$ for all $x \in M$ and $F(x) \leq \rho(x)$ for all $x \in X$.

NOTE: this implies the Hahn-Banach theorem, 18.5 on textbook p.173, in the special case $\rho(x) \stackrel{\text{def}}{=} \|x\|$.

22. Let X be a Banach space and suppose x and y are distinct points in X. Prove that there is a bounded linear functional f on X such that $f(x) \neq f(y)$.

Note: it may thus be said that there are enough bounded linear functionals on X to *separate points*.

23. Let X be a Banach space, let $A \subset X$ be an open convex set, and let $B \subset X$ be a convex set disjoint from A. Prove that there exists a bounded real-valued linear functional f and a constant $s \in \mathbf{R}$ such that $f(a) < s \leq f(b)$ for all $a \in A$ and all $b \in B$.

Hint: Consider the difference set $E = A - B + (a_0 - b_0)$ for fixed $a_0 \in A$, $b_0 \in B$, and apply exercises 18, 20, and 21.

24. Let X be a normed linear space. For any convex $B \subset X$, say that a subset $F \subset B$ is a *face* of B if, given $x, y \in B$ and $0 < \theta < 1$ with $\theta x + (1-\theta)y \in F$, one may conclude that $x, y \in F$.

   a. Suppose f is a bounded linear functional on X and $B \subset X$ is a convex subset such that $\beta \stackrel{\text{def}}{=} \sup\{f(x) : x \in B\}$ is finite. Define

$$F \stackrel{\text{def}}{=} \{x \in B : f(x) = \beta\}.$$

   Prove that F is a face of B.

   b. Suppose B is a convex set, $F \subset B$ is a face of B, and $G \subset F$ is any subset. Prove that G is a face of F if and only if G is a face of B.

25. Let X be a linear space. For any convex $B \subset X$, say that $e \in B$ is an *extreme point* of B iff

$$(\forall x, y \in B)(\forall \theta \in (0,1)) \quad e = \theta x + (1-\theta)y \;\Rightarrow\; e = x = y.$$

   a. Suppose B is an *open* convex set in a normed linear space X. Prove that B has no extreme points.

   b. Suppose B is a *compact* convex set in a Banach space X. Prove that if B is non-empty then B contains an extreme point.

   (Hint: Apply Zorn's lemma to the collection of closed non-empty faces of B partially ordered by $F_1 \le F_2$ iff F_2 is a face of F_1. Show that any maximal element contains a single point of B, which is therefore an extreme point.)

26. Let X be a linear space and $A \subset X$ any subset. Define the *convex hull* of A to be

$$\text{ch}\,(A) \stackrel{\text{def}}{=} \{\theta x + (1-\theta)y : x, y \in A; 0 \le \theta \le 1\}.$$

   If X is a normed linear space, define the *closed convex hull* of A to be the closure of $\text{ch}\,(A)$, and denote it by $\overline{\text{ch}}\,(A)$.

   a. Prove that if $A \subset B \subset X$, then $\overline{\text{ch}}\,(A) \subset \overline{\text{ch}}\,(B)$.

   b. Prove that if A is a closed convex set, then $A = \overline{\text{ch}}\,(A)$.

27. Let X be a Banach space and suppose that $A \subset X$ is compact and convex. Let $E \subset A$ be the set of extreme points of A as defined in exercise 25. Prove that $A = \overline{\text{ch}}\,(E)$.

   Hint: use exercises 23 and 26.

   Note: this is the *Krein-Milman* theorem.