Math 5052
Measure Theory and Functional Analysis II
Homework Assignment 8

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Due Friday, February 19th, 2016

Please do Exercises 2, 3, 5*, 6, 10*, 11, 12, 15, 16*, 17.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. Some of these exercises originate from that source.

1. Let $\langle f, g\rangle \stackrel{\text{def}}{=} \int_0^1 f(x)\overline{g(x)}\,dx$ be the usual inner product in $L^2([0,1])$. Prove that $C([0,1])$ is not a Hilbert space with respect to this inner product and its derived norm.

2. Suppose that $\{x_n\}$ is a sequence in a Hilbert space H. Suppose $\|x_n\| \to \|x\|$ and $(\forall y \in H)\, \langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$. Prove that $\|x_n - x\| \to 0$ as $n \to \infty$.

3. Suppose M is a subspace of a Hilbert space H.

   a. Prove that if M is closed, then $(M^\perp)^\perp = M$.

   b. Find a counterexample where M is not closed and $(M^\perp)^\perp \neq M$.

4. Prove that if H is infinite-dimensional, namely there are linearly independent subsets $\{x_1, \ldots, x_n\}$ for all $n = 1, 2, \ldots$, then the closed unit ball in H is not compact.

5. Suppose $\{a_n : n = 1, 2, \ldots\}$ is a sequence of real numbers such that

$$\sum_{n=1}^{\infty} a_n b_n < \infty$$

   whenever $\sum_{n=1}^{\infty} b_n^2 < \infty$. Prove that $\sum_{n=1}^{\infty} a_n^2 < \infty$.

6. Say that $x_n \to x$ *weakly* in a Hilbert space H if $\langle x_n, y\rangle \to \langle x, y\rangle$ for every $y \in H$. Prove that if x_n is a sequence in H bounded by $\sup_n \|x_n\| \leq 1$, then there is a subsequence n_j and an element $x \in H$ with $\|x\| \leq 1$ such that $x_{n_j} \to x$ weakly as $j \to \infty$.

7. If A is a Lebesgue measurable subset of $[0, 2\pi)$, prove that

$$\lim_{n\to\infty} \int_A e^{inx}\, dx = 0.$$

(This is a special case of the *Riemann-Lebesgue lemma.*)

8. Suppose that μ, ν are finite measures on a measurable space $(X, \mathcal{A})$, with $\nu \ll \mu$. Assume that $\nu(A) \le \mu(A)$ for all measurable A.

   For real-valued $f \in L^2(X, \mu)$, define $L(f) \stackrel{\text{def}}{=} \int_X f\, d\mu$.

   a. Show that L is a bounded linear functional on $L^2(X, \mu)$.

   b. Show that there exists a real-valued measurable function $g \in L^2(X, \mu)$ such that $L(f) = \int_X fg\, d\mu$ for all $f \in L^2(X, \mu)$. (Hint: use theorem 19.10 on textbook p.188.)

   c. Show that g from part b is the Radon-Nikodym derivative $d\nu/d\mu$.

   Note: by exercise 13.6 on textbook p.105, the assumption $\nu(A) \le \mu(A)$ imposes no restriction as one may replace μ with $\mu + \nu$. Hence, this is an alternate proof of the Radon-Nikodym theorem as a corollary of theorem 19.10.

9. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and *1-periodic*, namely $f(x + 1) = f(x)$ for all $x \in \mathbf{R}$. Prove that if $\gamma \in \mathbf{R}$ is irrational, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(j\gamma) = \int_0^1 f(x)\, dx.$$

(This is a special case of the *Birkhoff ergodic theorem.*)

10. Suppose that M is a closed subspace of a Hilbert space H. Fix $x \in H$ and define $x + M \stackrel{\text{def}}{=} \{x + y : y \in M\}$.

    a. Prove that $x + M$ is a closed convex subset of H.

    b. Let Qx be the (unique) point of $x + M$ with smallest norm and let $Px = x - Qx$. (P is called the *orthogonal projection* of x onto M.) Prove that P and Q are surjective from H to M and $M^\perp$, respectively.

    c. Prove that P and Q are linear mappings.

    d. Prove that if $y \in M$ then $Py = y$ and $Qy = 0$.

    e. Prove that if $z \in M^\perp$ then $Pz = 0$ and $Qz = z$.

    f. Prove that $\|w\|^2 = \|Pw\|^2 + \|Qw\|^2$ for any $w \in H$.

11. Suppose $\{e_n\}$ is a countable orthonormal basis for a Hilbert space H and $\{f_n\}$ is a countable orthonormal set such that

$$\sum_n \|e_n - f_n\|^2 < 1.$$

Prove that $\{f_n\}$ is a basis.

12. Suppose that $\{e_n\}$ and $\{f_n\}$ are countable orthonormal bases for a Hilbert space H. Define a linear transformation $T : H \to H$ by

$$T(\sum_n c_n e_n) = \sum_n c_n f_n.$$

   a. Prove that T is continuous and compute the operator norm of T.

   b. Prove that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in H$.

13. Let H, G be Hilbert spaces. Say that a linear function $T : H \to G$ is an *isometry* if $\langle Tx, Ty \rangle_G = \langle x, y \rangle_H$ for every $x, y \in H$.

   a. Prove that an isometry T is continuous and compute the operator norm of T.

   b. Prove that if T is an isometry then it is injective.

   c. Must an isometry be surjective? Supply a proof or a counterexample.

   d. Suppose T_1 and T_2 are isometries. Must $T_1 + T_2$ be an isometry?

14. Let H be a Hilbert spaces. Say that a linear function $T : H \to H$ is *selfadjoint* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for every $x \in H$.

   a. Prove that if T is selfadjoint, then $\langle Tx, x \rangle$ is real-valued for all $x \in H$.

   b. Say that selfajoint T is *positive definite* if $\langle Tx, x \rangle > 0$ for all $x \neq 0$. Prove that such T must be injective.

   c. Give an example of a positive definite T that is not surjective.

15. Let X, Y be Hilbert spaces. Suppose that $T : X \to Y$ is a bounded linear function. Prove that there exists a unique bounded linear function $T^* : Y \to X$ satisfying

$$(\forall x \in X)(\forall y \in Y)\ \ \langle Tx, y \rangle_Y = \langle x, T^* y \rangle_X .$$

(Such T^* is called the *adjoint* of T.)

16. Suppose X, Y are Hilbert spaces, and write $T^* : Y \to X$ for the adjoint of bounded linear function $T : X \to Y$ as in exercise 15.

   a. Prove that $(T^*)^* = T$. Conclude that $\|T\| = \|T^*\|$.

b. Show that $(T_1 + cT_2)^* = T_1^* + \bar{c}T_2^*$ for any T_1, T_2 and $c \in \mathbf{C}$.

c. Prove that $T^*T : X \to X$ is selfadjoint and, if T is injective, also positive definite.

d. Suppose that T is an isometry (see exercise 13). Prove that $T^*T : X \to X$ is the identity, and $TT^* : Y \to Y$ is an orthogonal projection.

e. Suppose that T is a *surjective* isometry. Prove that $TT^* : Y \to Y$ is the identity.

17. Suppose that H is a separable Hilbert space and write ℓ^2 for the Hilbert space of square-summable sequences in $\mathbf{C}$. Prove that there is a bijective linear isometry $T : \ell^2 \to H$. (Hint: see exercise 12.)

18. Define the *Haar function* $h : \mathbf{R} \to \mathbf{R}$ by

$$h \overset{\text{def}}{=} \chi_{[0,1/2)} - \chi_{[1/2,1)}; \qquad h(t) \overset{\text{def}}{=} \begin{cases} 0, & \text{if } t < 0 \text{ or } t \geq 1; \\ 1, & \text{if } 0 \leq t < 1/2; \\ -1, & \text{if } 1/2 \leq t < 1. \end{cases}$$

For integers j, k, define

$$h_{j,k}(t) \overset{\text{def}}{=} 2^{j/2} h(2^j t - k).$$

Prove that

$$H \overset{\text{def}}{=} \{h_{j,k} : j, k \in \mathbf{Z}\}$$

is an orthonormal basis for $L^2(\mathbf{R})$.

Note: H is called the *Haar basis.* It is defined by the *mother function* $h_{0,0} = h$, and its elements satisfy

$$\operatorname{supp} h_{j,k} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right].$$